On very non-linear subsets of continuous functions

Vinícius Vieira Fávaro

Symposium in Real Analysis XXXVII
São Carlos, 03-06 June 2013

Table of Contents

Introduction

- In this work we continue the study initiated by Gurariy and Quarta in 2004 on the existence of linear spaces formed, up to the null vector, by continuous functions that attain the maximum only at one point.
- The problem of finding linear spaces formed, up to the origin, solely by real-valued continuous functions on certain subsets of $\mathbb{R}$ that attain the maximum only at one point was successfully investigated by V.I. Gurariy and L. Quarta [1].
- Given a topological space D, by $\widehat{C}(D)$ we denote the subset of the linear space $C(D)$ of all real-valued continuous functions on D composed by the functions that attain the maximum exactly once in D. The main results obtained by Gurariy and Quarta in this direction are the following:

Introduction

- In this work we continue the study initiated by Gurariy and Quarta in 2004 on the existence of linear spaces formed, up to the null vector, by continuous functions that attain the maximum only at one point.
- The problem of finding linear spaces formed, up to the origin, solely by real-valued continuous functions on certain subsets of $\mathbb{R}$ that attain the maximum only at one point was successfully investigated by V.I. Gurariy and L. Quarta [1].
- Given a topological space D, by $\widehat{C}(D)$ we denote the subset of the linear space $C(D)$ of all real-valued continuous functions on D composed by the functions that attain the maximum exactly once in D. The main results obtained by Gurariy and Quarta in this direction are the following:

Introduction

- In this work we continue the study initiated by Gurariy and Quarta in 2004 on the existence of linear spaces formed, up to the null vector, by continuous functions that attain the maximum only at one point.
- The problem of finding linear spaces formed, up to the origin, solely by real-valued continuous functions on certain subsets of $\mathbb{R}$ that attain the maximum only at one point was successfully investigated by V.I. Gurariy and L. Quarta [1].
- Given a topological space D, by $\widehat{C}(D)$ we denote the subset of the linear space $C(D)$ of all real-valued continuous functions on D composed by the functions that attain the maximum exactly once in D. The main results obtained by Gurariy and Quarta in this direction are the following:

Introduction

- In this work we continue the study initiated by Gurariy and Quarta in 2004 on the existence of linear spaces formed, up to the null vector, by continuous functions that attain the maximum only at one point.
- The problem of finding linear spaces formed, up to the origin, solely by real-valued continuous functions on certain subsets of $\mathbb{R}$ that attain the maximum only at one point was successfully investigated by V.I. Gurariy and L. Quarta [1].
- Given a topological space D, by $\widehat{C}(D)$ we denote the subset of the linear space $C(D)$ of all real-valued continuous functions on D composed by the functions that attain the maximum exactly once in D. The main results obtained by Gurariy and Quarta in this direction are the following:

(A) $\widehat{C}[a,b)$ contains, up to the origin, a 2-dimensional linear subspace of $C[a,b)$.

(B) $\widehat{C}(\mathbb{R})$ contains, up to the origin, a 2-dimensional linear subspace of $C(\mathbb{R})$.

(C) There is no 2-dimensional linear subspace of $C[a,b]$ contained in $\widehat{C}[a,b] \cup \{0\}$.

- In the words of Gurariy and Quarta (see, [1]): $\widehat{C}[a,b)$ and $\widehat{C}(\mathbb{R})$ are 2-lineable and $\widehat{C}[a,b]$ is very non-linear.
- We remember that a subset A of a vector space E is *μ-lineable*, where μ is a cardinal number, if $A \cup \{0\}$ contains a μ-dimensional subspace of E.
- There are several papers devoted to the search for linear spaces enjoying certain *special* properties (see, for example, the Plenary Talk of D. Pellegrino on Thursday morning!!).

(A) $\widehat{C}[a,b)$ contains, up to the origin, a 2-dimensional linear subspace of $C[a,b)$.

(B) $\widehat{C}(\mathbb{R})$ contains, up to the origin, a 2-dimensional linear subspace of $C(\mathbb{R})$.

(C) There is no 2-dimensional linear subspace of $C[a,b]$ contained in $\widehat{C}[a,b] \cup \{0\}$.

- In the words of Gurariy and Quarta (see, [1]): $\widehat{C}[a,b)$ and $\widehat{C}(\mathbb{R})$ are 2-lineable and $\widehat{C}[a,b]$ is very non-linear.
- We remember that a subset A of a vector space E is *μ-lineable*, where μ is a cardinal number, if $A \cup \{0\}$ contains a μ-dimensional subspace of E.
- There are several papers devoted to the search for linear spaces enjoying certain *special* properties (see, for example, the Plenary Talk of D. Pellegrino on Thursday morning!!).

(A) $\widehat{C}[a,b)$ contains, up to the origin, a 2-dimensional linear subspace of $C[a,b)$.

(B) $\widehat{C}(\mathbb{R})$ contains, up to the origin, a 2-dimensional linear subspace of $C(\mathbb{R})$.

(C) There is no 2-dimensional linear subspace of $C\,[a,b]$ contained in $\widehat{C}\,[a,b] \cup \{0\}$.

- In the words of Gurariy and Quarta (see, [1]): $\widehat{C}[a,b)$ and $\widehat{C}(\mathbb{R})$ are 2-lineable and $\widehat{C}\,[a,b]$ is very non-linear.
- We remember that a subset A of a vector space E is *μ-lineable*, where μ is a cardinal number, if $A \cup \{0\}$ contains a μ-dimensional subspace of E.
- There are several papers devoted to the search for linear spaces enjoying certain *special* properties (see, for example, the Plenary Talk of D. Pellegrino on Thursday morning!!).

(A) $\widehat{C}[a,b)$ contains, up to the origin, a 2-dimensional linear subspace of $C[a,b)$.

(B) $\widehat{C}(\mathbb{R})$ contains, up to the origin, a 2-dimensional linear subspace of $C(\mathbb{R})$.

(C) There is no 2-dimensional linear subspace of $C\,[a,b]$ contained in $\widehat{C}\,[a,b] \cup \{0\}$.

- In the words of Gurariy and Quarta (see, [1]): $\widehat{C}[a,b)$ and $\widehat{C}(\mathbb{R})$ are 2-lineable and $\widehat{C}\,[a,b]$ is very non-linear.
- We remember that a subset A of a vector space E is *μ-lineable*, where μ is a cardinal number, if $A \cup \{0\}$ contains a μ-dimensional subspace of E.
- There are several papers devoted to the search for linear spaces enjoying certain *special* properties (see, for example, the Plenary Talk of D. Pellegrino on Thursday morning!!).

(A) $\widehat{C}[a,b)$ contains, up to the origin, a 2-dimensional linear subspace of $C[a,b)$.

(B) $\widehat{C}(\mathbb{R})$ contains, up to the origin, a 2-dimensional linear subspace of $C(\mathbb{R})$.

(C) There is no 2-dimensional linear subspace of $C\,[a,b]$ contained in $\widehat{C}\,[a,b] \cup \{0\}$.

- In the words of Gurariy and Quarta (see, [1]): $\widehat{C}[a,b)$ and $\widehat{C}(\mathbb{R})$ are 2-lineable and $\widehat{C}\,[a,b]$ is very non-linear.
- We remember that a subset A of a vector space E is μ-*lineable*, where μ is a cardinal number, if $A \cup \{0\}$ contains a μ-dimensional subspace of E.
- There are several papers devoted to the search for linear spaces enjoying certain *special* properties (see, for example, the Plenary Talk of D. Pellegrino on Thursday morning!!).

(A) $\widehat{C}[a,b)$ contains, up to the origin, a 2-dimensional linear subspace of $C[a,b)$.

(B) $\widehat{C}(\mathbb{R})$ contains, up to the origin, a 2-dimensional linear subspace of $C(\mathbb{R})$.

(C) There is no 2-dimensional linear subspace of $C\,[a,b]$ contained in $\widehat{C}\,[a,b] \cup \{0\}$.

- In the words of Gurariy and Quarta (see, [1]): $\widehat{C}[a,b)$ and $\widehat{C}(\mathbb{R})$ are 2-lineable and $\widehat{C}\,[a,b]$ is very non-linear.
- We remember that a subset A of a vector space E is *μ-lineable*, where μ is a cardinal number, if $A \cup \{0\}$ contains a μ-dimensional subspace of E.
- There are several papers devoted to the search for linear spaces enjoying certain *special* properties (see, for example, the Plenary Talk of D. Pellegrino on Thursday morning!!).

(A) $\widehat{C}[a,b)$ contains, up to the origin, a 2-dimensional linear subspace of $C[a,b)$.

(B) $\widehat{C}(\mathbb{R})$ contains, up to the origin, a 2-dimensional linear subspace of $C(\mathbb{R})$.

(C) There is no 2-dimensional linear subspace of $C\,[a,b]$ contained in $\widehat{C}\,[a,b] \cup \{0\}$.

- In the words of Gurariy and Quarta (see, [1]): $\widehat{C}[a,b)$ and $\widehat{C}(\mathbb{R})$ are 2-lineable and $\widehat{C}\,[a,b]$ is very non-linear.
- We remember that a subset A of a vector space E is μ-*lineable*, where μ is a cardinal number, if $A \cup \{0\}$ contains a μ-dimensional subspace of E.
- There are several papers devoted to the search for linear spaces enjoying certain *special* properties (see, for example, the Plenary Talk of D. Pellegrino on Thursday morning!!).

The purpose of this work is to obtain far-reaching generalizations of the aforementioned results of Gurariy and Quarta:

- We extend (A) to spaces of functions defined on topological spaces D that can be continuously embedded onto some Euclidean sphere S^n.
- We extend (B) to spaces of functions defined on quite general topological spaces D that include $\mathbb{R}$.
- We extend (C) to spaces of functions defined on compact subsets K of $\mathbb{R}^m$. In this case we prove that $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$ for every compact $K \subset \mathbb{R}^m$ but, on the other hand, there are compact sets $K \subset \mathbb{R}^m$ for which $\widehat{C}(K) \cup \{0\}$ contains an m-dimensional subspace of $C(K)$.

The purpose of this work is to obtain far-reaching generalizations of the aforementioned results of Gurariy and Quarta:

- We extend (A) to spaces of functions defined on topological spaces D that can be continuously embedded onto some Euclidean sphere S^n.
- We extend (B) to spaces of functions defined on quite general topological spaces D that include $\mathbb{R}$.
- We extend (C) to spaces of functions defined on compact subsets K of $\mathbb{R}^m$. In this case we prove that $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$ for every compact $K \subset \mathbb{R}^m$ but, on the other hand, there are compact sets $K \subset \mathbb{R}^m$ for which $\widehat{C}(K) \cup \{0\}$ contains an m-dimensional subspace of $C(K)$.

The purpose of this work is to obtain far-reaching generalizations of the aforementioned results of Gurariy and Quarta:

- We extend (A) to spaces of functions defined on topological spaces D that can be continuously embedded onto some Euclidean sphere S^n.
- We extend (B) to spaces of functions defined on quite general topological spaces D that include $\mathbb{R}$.
- We extend (C) to spaces of functions defined on compact subsets K of $\mathbb{R}^m$. In this case we prove that $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$ for every compact $K \subset \mathbb{R}^m$ but, on the other hand, there are compact sets $K \subset \mathbb{R}^m$ for which $\widehat{C}(K) \cup \{0\}$ contains an m-dimensional subspace of $C(K)$.

The purpose of this work is to obtain far-reaching generalizations of the aforementioned results of Gurariy and Quarta:

- We extend (A) to spaces of functions defined on topological spaces D that can be continuously embedded onto some Euclidean sphere S^n.
- We extend (B) to spaces of functions defined on quite general topological spaces D that include $\mathbb{R}$.
- We extend (C) to spaces of functions defined on compact subsets K of $\mathbb{R}^m$. In this case we prove that $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$ for every compact $K \subset \mathbb{R}^m$ but, on the other hand, there are compact sets $K \subset \mathbb{R}^m$ for which $\widehat{C}(K) \cup \{0\}$ contains an m-dimensional subspace of $C(K)$.

The purpose of this work is to obtain far-reaching generalizations of the aforementioned results of Gurariy and Quarta:

- We extend (A) to spaces of functions defined on topological spaces D that can be continuously embedded onto some Euclidean sphere S^n.
- We extend (B) to spaces of functions defined on quite general topological spaces D that include $\mathbb{R}$.
- We extend (C) to spaces of functions defined on compact subsets K of $\mathbb{R}^m$. In this case we prove that $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$ for every compact $K \subset \mathbb{R}^m$ but, on the other hand, there are compact sets $K \subset \mathbb{R}^m$ for which $\widehat{C}(K) \cup \{0\}$ contains an m-dimensional subspace of $C(K)$.

The purpose of this work is to obtain far-reaching generalizations of the aforementioned results of Gurariy and Quarta:

- We extend (A) to spaces of functions defined on topological spaces D that can be continuously embedded onto some Euclidean sphere S^n.
- We extend (B) to spaces of functions defined on quite general topological spaces D that include $\mathbb{R}$.
- We extend (C) to spaces of functions defined on compact subsets K of $\mathbb{R}^m$. In this case we prove that $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$ for every compact $K \subset \mathbb{R}^m$ but, on the other hand, there are compact sets $K \subset \mathbb{R}^m$ for which $\widehat{C}(K) \cup \{0\}$ contains an m-dimensional subspace of $C(K)$.

Generalizing (A)

Now we show that in (A) the interval $[a, b)$ can be replaced by preimages D of Euclidean spheres. Moreover, the dimension of the resulting subspace of $C(D)$ contained in $\widehat{C}(D) \cup \{0\}$ equals the dimension of the sphere plus 1. By $\langle \cdot, \cdot \rangle$ we denote the usual inner product in the Euclidean spaces.

Theorem

Let $n \geq 2$ be a positive integer and D be a topological space for which there is a continuous bijection from D to S^{n-1}. Then $\widehat{C}(D)$ contains, up to the origin, an n-dimensional linear subspace of $C(D)$.

Generalizing (A)

Now we show that in (A) the interval $[a, b)$ can be replaced by preimages D of Euclidean spheres. Moreover, the dimension of the resulting subspace of $C(D)$ contained in $\widehat{C}(D) \cup \{0\}$ equals the dimension of the sphere plus 1. By $\langle \cdot, \cdot \rangle$ we denote the usual inner product in the Euclidean spaces.

Theorem

Let $n \geq 2$ be a positive integer and D be a topological space for which there is a continuous bijection from D to S^{n-1}. Then $\widehat{C}(D)$ contains, up to the origin, an n-dimensional linear subspace of $C(D)$.

Generalizing (A)

Now we show that in (A) the interval $[a, b)$ can be replaced by preimages D of Euclidean spheres. Moreover, the dimension of the resulting subspace of $C(D)$ contained in $\widehat{C}(D) \cup \{0\}$ equals the dimension of the sphere plus 1. By $\langle \cdot, \cdot \rangle$ we denote the usual inner product in the Euclidean spaces.

Theorem

Let $n \geq 2$ be a positive integer and D be a topological space for which there is a continuous bijection from D to S^{n-1}. Then $\widehat{C}(D)$ contains, up to the origin, an n-dimensional linear subspace of $C(D)$.

Generalizing (A)

Now we show that in (A) the interval $[a, b)$ can be replaced by preimages D of Euclidean spheres. Moreover, the dimension of the resulting subspace of $C(D)$ contained in $\widehat{C}(D) \cup \{0\}$ equals the dimension of the sphere plus 1. By $\langle \cdot, \cdot \rangle$ we denote the usual inner product in the Euclidean spaces.

Theorem

Let $n \geq 2$ be a positive integer and D be a topological space for which there is a continuous bijection from D to S^{n-1}. Then $\widehat{C}(D)$ contains, up to the origin, an n-dimensional linear subspace of $C(D)$.

Sketch of the proof:

Let $\pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ be the projection on the component $i = 1, \ldots, n$ and let $G \colon D \longrightarrow S^{n-1}$ be a continuous bijection.

Step 1: The maps π_i, $1 \leq i \leq n$, are linearly independent.
Straightforward.

Step 2: Each non-trivial linear combination of the functions π_i, $1 \leq i \leq n$, has only one point of maximum.
Consider $\sum_{i=1}^{n} a_i \pi_i$ with some $a_i \neq 0$. For an arbitrary $y \in S^{n-1}$,

$$\sum_{i=1}^{n} a_i \pi_i(y) = \langle a, y \rangle,$$

where $a = (a_1, \ldots, a_n)$. It is well known that the function $y \mapsto \langle a, y \rangle$ attains its maximum in $z \in S^{n-1}$ if, and only if,

Sketch of the proof:

Let $\pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ be the projection on the component $i = 1, \dots, n$ and let $G \colon D \longrightarrow S^{n-1}$ be a continuous bijection.

Step 1: The maps π_i, $1 \leq i \leq n$, are linearly independent.
Straightforward.

Step 2: Each non-trivial linear combination of the functions π_i, $1 \leq i \leq n$, has only one point of maximum.
Consider $\sum_{i=1}^{n} a_i \pi_i$ with some $a_i \neq 0$. For an arbitrary $y \in S^{n-1}$,

$$\sum_{i=1}^{n} a_i \pi_i(y) = \langle a, y \rangle,$$

where $a = (a_1, \dots, a_n)$. It is well known that the function $y \mapsto \langle a, y \rangle$ attains its maximum in $z \in S^{n-1}$ if, and only if,

Sketch of the proof:

Let $\pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ be the projection on the component $i = 1, \ldots, n$ and let $G \colon D \longrightarrow S^{n-1}$ be a continuous bijection.

Step 1: The maps π_i, $1 \leq i \leq n$, are linearly independent.

Straightforward.

Step 2: Each non-trivial linear combination of the functions π_i, $1 \leq i \leq n$, has only one point of maximum.

Consider $\sum_{i=1}^{n} a_i \pi_i$ with some $a_i \neq 0$. For an arbitrary $y \in S^{n-1}$,

$$\sum_{i=1}^{n} a_i \pi_i(y) = \langle a, y \rangle,$$

where $a = (a_1, \ldots, a_n)$. It is well known that the function $y \mapsto \langle a, y \rangle$ attains its maximum in $z \in S^{n-1}$ if, and only if,

Sketch of the proof:

Let $\pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ be the projection on the component $i = 1, \ldots, n$ and let $G \colon D \longrightarrow S^{n-1}$ be a continuous bijection.

Step 1: The maps π_i, $1 \leq i \leq n$, are linearly independent.
Straightforward.

Step 2: Each non-trivial linear combination of the functions π_i, $1 \leq i \leq n$, has only one point of maximum.
Consider $\sum_{i=1}^{n} a_i \pi_i$ with some $a_i \neq 0$. For an arbitrary $y \in S^{n-1}$,

$$\sum_{i=1}^{n} a_i \pi_i(y) = \langle a, y \rangle,$$

where $a = (a_1, \ldots, a_n)$. It is well known that the function $y \mapsto \langle a, y \rangle$ attains its maximum in $z \in S^{n-1}$ if, and only if,

Sketch of the proof:

Let $\pi_i\colon S^{n-1} \longrightarrow \mathbb{R}$ be the projection on the component $i = 1, \ldots, n$ and let $G\colon D \longrightarrow S^{n-1}$ be a continuous bijection.
Step 1: The maps π_i, $1 \le i \le n$, are linearly independent.
Straightforward.

Step 2: Each non-trivial linear combination of the functions π_i, $1 \le i \le n$, has only one point of maximum.
Consider $\sum_{i=1}^{n} a_i \pi_i$ with some $a_i \neq 0$. For an arbitrary $y \in S^{n-1}$,

$$\sum_{i=1}^{n} a_i \pi_i(y) = \langle a, y \rangle,$$

where $a = (a_1, \ldots, a_n)$. It is well known that the function $y \mapsto \langle a, y \rangle$ attains its maximum in $z \in S^{n-1}$ if, and only if,

Sketch of the proof:

Let $\pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ be the projection on the component $i = 1, \ldots, n$ and let $G \colon D \longrightarrow S^{n-1}$ be a continuous bijection.
Step 1: The maps π_i, $1 \leq i \leq n$, are linearly independent. Straightforward.

Step 2: Each non-trivial linear combination of the functions π_i, $1 \leq i \leq n$, has only one point of maximum. Consider $\sum_{i=1}^{n} a_i \pi_i$ with some $a_i \neq 0$. For an arbitrary $y \in S^{n-1}$,

$$\sum_{i=1}^{n} a_i \pi_i(y) = \langle a, y \rangle,$$

where $a = (a_1, \ldots, a_n)$. It is well known that the function $y \mapsto \langle a, y \rangle$ attains its maximum in $z \in S^{n-1}$ if, and only if,

Sketch of the proof:

Let $\pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ be the projection on the component $i = 1, \ldots, n$ and let $G \colon D \longrightarrow S^{n-1}$ be a continuous bijection.

Step 1: The maps π_i, $1 \leq i \leq n$, are linearly independent.
Straightforward.

Step 2: Each non-trivial linear combination of the functions π_i, $1 \leq i \leq n$, has only one point of maximum.
Consider $\sum_{i=1}^{n} a_i \pi_i$ with some $a_i \neq 0$. For an arbitrary $y \in S^{n-1}$,

$$\sum_{i=1}^{n} a_i \pi_i(y) = \langle a, y \rangle,$$

where $a = (a_1, \ldots, a_n)$. It is well known that the function $y \mapsto \langle a, y \rangle$ attains its maximum in $z \in S^{n-1}$ if, and only if,

Sketch of the proof:

Let $\pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ be the projection on the component $i = 1, \ldots, n$ and let $G \colon D \longrightarrow S^{n-1}$ be a continuous bijection.

Step 1: The maps π_i, $1 \leq i \leq n$, are linearly independent. Straightforward.

Step 2: Each non-trivial linear combination of the functions π_i, $1 \leq i \leq n$, has only one point of maximum. Consider $\sum_{i=1}^{n} a_i \pi_i$ with some $a_i \neq 0$. For an arbitrary $y \in S^{n-1}$,

$$\sum_{i=1}^{n} a_i \pi_i(y) = \langle a, y \rangle,$$

where $a = (a_1, \ldots, a_n)$. It is well known that the function $y \mapsto \langle a, y \rangle$ attains its maximum in $z \in S^{n-1}$ if, and only if,

Sketch of the proof:

$$z = \frac{(a_1, \ldots, a_n)}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}}. \tag{1}$$

Therefore the unique point of maximum of $\sum_{i=1}^{n} a_i \pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ is z as in (1).

Step 3: Construction of the n-dimensional vector space. Consider the compositions $\pi_i \circ G \colon D \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$. It is easy to see that set $\{\pi_i \circ G : i = 1, \ldots, n\}$ is linearly independent. Let $h = \sum_{i=1}^{n} b_i (\pi_i \circ G)$ be a non-trivial linear combination of the functions $\pi_i \circ G$. Since $\sum_{i=1}^{n} b_i \pi_i$ attains its maximum at a unique point $x_0 \in S^{n-1}$, and G is a bijection, it follows that h attains its maximum at an unique point, namely, $G^{-1}(x_0)$. The linear subspace spanned by the functions $\pi_i \circ G, i = 1, \ldots, n$, completes the proof.□

Sketch of the proof:

$$z = \frac{(a_1, \ldots, a_n)}{\left(\sum_{i=1}^n a_i^2\right)^{1/2}}. \tag{1}$$

Therefore the unique point of maximum of $\sum_{i=1}^n a_i \pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ is z as in (1).

Step 3: Construction of the n-dimensional vector space. Consider the compositions $\pi_i \circ G \colon D \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$. It is easy to see that set $\{\pi_i \circ G : i = 1, \ldots, n\}$ is linearly independent. Let $h = \sum_{i=1}^n b_i (\pi_i \circ G)$ be a non-trivial linear combination of the functions $\pi_i \circ G$. Since $\sum_{i=1}^n b_i \pi_i$ attains its maximum at a unique point $x_0 \in S^{n-1}$, and G is a bijection, it follows that h attains its maximum at an unique point, namely, $G^{-1}(x_0)$. The linear subspace spanned by the functions $\pi_i \circ G, i = 1, \ldots, n$, completes the proof.$\square$

Sketch of the proof:

$$z = \frac{(a_1, \ldots, a_n)}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}}. \tag{1}$$

Therefore the unique point of maximum of $\sum_{i=1}^{n} a_i \pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ is z as in (1).

Step 3: Construction of the n-dimensional vector space.

Consider the compositions $\pi_i \circ G \colon D \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$. It is easy to see that set $\{\pi_i \circ G : i = 1, \ldots, n\}$ is linearly independent. Let $h = \sum_{i=1}^{n} b_i (\pi_i \circ G)$ be a non-trivial linear combination of the functions $\pi_i \circ G$. Since $\sum_{i=1}^{n} b_i \pi_i$ attains its maximum at a unique point $x_0 \in S^{n-1}$, and G is a bijection, it follows that h attains its maximum at an unique point, namely, $G^{-1}(x_0)$. The linear subspace spanned by the functions $\pi_i \circ G, i = 1, \ldots, n$, completes the proof.□

Sketch of the proof:

$$z = \frac{(a_1, \ldots, a_n)}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}}. \tag{1}$$

Therefore the unique point of maximum of $\sum_{i=1}^{n} a_i \pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ is z as in (1).

Step 3: Construction of the n-dimensional vector space. Consider the compositions $\pi_i \circ G \colon D \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$. It is easy to see that set $\{\pi_i \circ G : i = 1, \ldots, n\}$ is linearly independent. Let $h = \sum_{i=1}^{n} b_i (\pi_i \circ G)$ be a non-trivial linear combination of the functions $\pi_i \circ G$. Since $\sum_{i=1}^{n} b_i \pi_i$ attains its maximum at a unique point $x_0 \in S^{n-1}$, and G is a bijection, it follows that h attains its maximum at an unique point, namely, $G^{-1}(x_0)$. The linear subspace spanned by the functions $\pi_i \circ G, i = 1, \ldots, n$, completes the proof.$\square$

Sketch of the proof:

$$z = \frac{(a_1, \ldots, a_n)}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}}. \tag{1}$$

Therefore the unique point of maximum of $\sum_{i=1}^{n} a_i \pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ is z as in (1).

Step 3: Construction of the n-dimensional vector space. Consider the compositions $\pi_i \circ G \colon D \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$. It is easy to see that set $\{\pi_i \circ G : i = 1, \ldots, n\}$ is linearly independent. Let $h = \sum_{i=1}^{n} b_i (\pi_i \circ G)$ be a non-trivial linear combination of the functions $\pi_i \circ G$. Since $\sum_{i=1}^{n} b_i \pi_i$ attains its maximum at a unique point $x_0 \in S^{n-1}$, and G is a bijection, it follows that h attains its maximum at an unique point, namely, $G^{-1}(x_0)$. The linear subspace spanned by the functions $\pi_i \circ G, i = 1, \ldots, n$, completes the proof.□

Sketch of the proof:

$$z = \frac{(a_1, \ldots, a_n)}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}}. \tag{1}$$

Therefore the unique point of maximum of $\sum_{i=1}^{n} a_i \pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ is z as in (1).

Step 3: Construction of the n-dimensional vector space. Consider the compositions $\pi_i \circ G \colon D \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$. It is easy to see that set $\{\pi_i \circ G : i = 1, \ldots, n\}$ is linearly independent. Let $h = \sum_{i=1}^{n} b_i (\pi_i \circ G)$ be a non-trivial linear combination of the functions $\pi_i \circ G$. Since $\sum_{i=1}^{n} b_i \pi_i$ attains its maximum at a unique point $x_0 \in S^{n-1}$, and G is a bijection, it follows that h attains its maximum at an unique point, namely, $G^{-1}(x_0)$. The linear subspace spanned by the functions $\pi_i \circ G, i = 1, \ldots, n$, completes the proof.□

Sketch of the proof:

$$z = \frac{(a_1, \dots, a_n)}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}}. \tag{1}$$

Therefore the unique point of maximum of $\sum_{i=1}^{n} a_i \pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ is z as in (1).

Step 3: Construction of the n-dimensional vector space. Consider the compositions $\pi_i \circ G \colon D \longrightarrow \mathbb{R}$ for $i = 1, \dots, n$. It is easy to see that set $\{\pi_i \circ G : i = 1, \dots, n\}$ is linearly independent. Let $h = \sum_{i=1}^{n} b_i (\pi_i \circ G)$ be a non-trivial linear combination of the functions $\pi_i \circ G$. Since $\sum_{i=1}^{n} b_i \pi_i$ attains its maximum at a unique point $x_0 \in S^{n-1}$, and G is a bijection, it follows that h attains its maximum at an unique point, namely, $G^{-1}(x_0)$. The linear subspace spanned by the functions $\pi_i \circ G, i = 1, \dots, n$, completes the proof.□

Sketch of the proof:

$$z = \frac{(a_1, \ldots, a_n)}{\left(\sum_{i=1}^n a_i^2\right)^{1/2}}. \qquad (1)$$

Therefore the unique point of maximum of $\sum_{i=1}^n a_i \pi_i \colon S^{n-1} \longrightarrow \mathbb{R}$ is z as in (1).

Step 3: Construction of the n-dimensional vector space. Consider the compositions $\pi_i \circ G \colon D \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$. It is easy to see that set $\{\pi_i \circ G : i = 1, \ldots, n\}$ is linearly independent. Let $h = \sum_{i=1}^n b_i (\pi_i \circ G)$ be a non-trivial linear combination of the functions $\pi_i \circ G$. Since $\sum_{i=1}^n b_i \pi_i$ attains its maximum at a unique point $x_0 \in S^{n-1}$, and G is a bijection, it follows that h attains its maximum at an unique point, namely, $G^{-1}(x_0)$. The linear subspace spanned by the functions $\pi_i \circ G, i = 1, \ldots, n$, completes the proof.□

Generalizing (B)

Now we show that the argument of the proof of Theorem 1 actually holds for more general domains; general enough to have $\mathbb{R}$ as a particular instance. By $\|\cdot\|_2$ we mean the Euclidean norm on $\mathbb{R}^n$.

Theorem

Let $n \geq 2$ be a positive integer and D be a topological space containing a closed set Y such that there are a continuous bijection $F\colon Y \longrightarrow S^{n-1}$ and a continuous extension $G\colon D \longrightarrow \mathbb{R}^n$ of F such that $\|G(x)\|_2 < 1$ for every $x \notin Y$. Then $\widehat{C}(D)$ contains, up to the origin, an n-dimensional linear subspace of $C(D)$.

Generalizing (B)

Now we show that the argument of the proof of Theorem 1 actually holds for more general domains; general enough to have $\mathbb{R}$ as a particular instance. By $\|\cdot\|_2$ we mean the Euclidean norm on $\mathbb{R}^n$.

Theorem

Let $n \geq 2$ be a positive integer and D be a topological space containing a closed set Y such that there are a continuous bijection $F\colon Y \longrightarrow S^{n-1}$ and a continuous extension $G\colon D \longrightarrow \mathbb{R}^n$ of F such that $\|G(x)\|_2 < 1$ for every $x \notin Y$. Then $\widehat{C}(D)$ contains, up to the origin, an n-dimensional linear subspace of $C(D)$.

Generalizing (B)

Now we show that the argument of the proof of Theorem 1 actually holds for more general domains; general enough to have $\mathbb{R}$ as a particular instance. By $\|\cdot\|_2$ we mean the Euclidean norm on $\mathbb{R}^n$.

Theorem

Let $n \geq 2$ be a positive integer and D be a topological space containing a closed set Y such that there are a continuous bijection $F\colon Y \longrightarrow S^{n-1}$ and a continuous extension $G\colon D \longrightarrow \mathbb{R}^n$ of F such that $\|G(x)\|_2 < 1$ for every $x \notin Y$. Then $\widehat{C}\,(D)$ contains, up to the origin, an n-dimensional linear subspace of $C(D)$.

Sketch of the proof:

The symbol π_i stands for the i-th projection on $\mathbb{R}^n$. Note that $S^{n-1} \subset G(D)$ and $G(D)$ is contained in the closed unit ball of $\mathbb{R}^n$.

Step 1: Any non-trivial linear combination $f := \sum_{i=1}^n b_i\pi_i : G(D) \longrightarrow \mathbb{R}$ attains its maximum at an unique point $x_0 \in G(D)$ and this point belongs to S^{n-1}.

Restricting $\sum_{i=1}^n b_i\pi_i$ to $S^{n-1} \subset G(D)$, the same argument of the previous section tells us that there is an unique $x_0 \in S^{n-1}$ such that

$$\sum_{i=1}^n b_i\pi_i(x) \le \sum_{i=1}^n b_i\pi_i(x_0) \qquad (2)$$

for every $x \in S^{n-1}$. Now we just need to show that if $y \in G(D) - S^{n-1}$, then

$$\sum_{i=1}^n b_i\pi_i(y) < \sum_{i=1}^n b_i\pi_i(x_0).$$

Sketch of the proof:

The symbol π_i stands for the i-th projection on $\mathbb{R}^n$. Note that $S^{n-1} \subset G(D)$ and $G(D)$ is contained in the closed unit ball of $\mathbb{R}^n$.

Step 1: Any non-trivial linear combination $f := \sum_{i=1}^n b_i\pi_i \colon G(D) \longrightarrow \mathbb{R}$ attains its maximum at an unique point $x_0 \in G(D)$ and this point belongs to S^{n-1}.

Restricting $\sum_{i=1}^n b_i\pi_i$ to $S^{n-1} \subset G(D)$, the same argument of the previous section tells us that there is an unique $x_0 \in S^{n-1}$ such that

$$\sum_{i=1}^{n} b_i\pi_i(x) \leq \sum_{i=1}^{n} b_i\pi_i(x_0) \tag{2}$$

for every $x \in S^{n-1}$. Now we just need to show that if $y \in G(D) - S^{n-1}$, then

$$\sum_{i=1}^{n} b_i\pi_i(y) < \sum_{i=1}^{n} b_i\pi_i(x_0).$$

Sketch of the proof:

The symbol π_i stands for the i-th projection on $\mathbb{R}^n$. Note that $S^{n-1} \subset G(D)$ and $G(D)$ is contained in the closed unit ball of $\mathbb{R}^n$.

Step 1: Any non-trivial linear combination $f := \sum_{i=1}^n b_i\pi_i : G(D) \longrightarrow \mathbb{R}$ attains its maximum at an unique point $x_0 \in G(D)$ and this point belongs to S^{n-1}.

Restricting $\sum_{i=1}^n b_i\pi_i$ to $S^{n-1} \subset G(D)$, the same argument of the previous section tells us that there is an unique $x_0 \in S^{n-1}$ such that

$$\sum_{i=1}^n b_i\pi_i(x) \leq \sum_{i=1}^n b_i\pi_i(x_0) \tag{2}$$

for every $x \in S^{n-1}$. Now we just need to show that if $y \in G(D) - S^{n-1}$, then

$$\sum_{i=1}^n b_i\pi_i(y) < \sum_{i=1}^n b_i\pi_i(x_0).$$

Sketch of the proof:

The symbol π_i stands for the i-th projection on $\mathbb{R}^n$. Note that $S^{n-1} \subset G(D)$ and $G(D)$ is contained in the closed unit ball of $\mathbb{R}^n$.

Step 1: Any non-trivial linear combination $f := \sum_{i=1}^n b_i\pi_i : G(D) \longrightarrow \mathbb{R}$ attains its maximum at an unique point $x_0 \in G(D)$ and this point belongs to S^{n-1}.

Restricting $\sum_{i=1}^n b_i\pi_i$ to $S^{n-1} \subset G(D)$, the same argument of the previous section tells us that there is an unique $x_0 \in S^{n-1}$ such that

$$\sum_{i=1}^n b_i\pi_i(x) \leq \sum_{i=1}^n b_i\pi_i(x_0) \qquad (2)$$

for every $x \in S^{n-1}$. Now we just need to show that if $y \in G(D) - S^{n-1}$, then

$$\sum_{i=1}^n b_i\pi_i(y) < \sum_{i=1}^n b_i\pi_i(x_0).$$

Sketch of the proof:

Using that $\|G(x)\|_2 < 1$, when $G(x) \in G(D) - S^{n-1}$, we obtain the aforementioned inequality.

Step 2: Any non-trivial linear combination

$$h := \sum_{i=1}^{n} b_i \left(\pi_i \circ G\right) : D \longrightarrow \mathbb{R}$$

attains its maximum only at the point $F^{-1}(x_0) \in S^{n-1}$. Not difficult!

Sketch of the proof:

Using that $\|G(x)\|_2 < 1$, when $G(x) \in G(D) - S^{n-1}$, we obtain the aforementioned inequality.

Step 2: Any non-trivial linear combination

$$h := \sum_{i=1}^{n} b_i \left(\pi_i \circ G\right) : D \longrightarrow \mathbb{R}$$

attains its maximum only at the point $F^{-1}\left(x_0\right) \in S^{n-1}$.

Not difficult!

Sketch of the proof:

Using that $\|G(x)\|_2 < 1$, when $G(x) \in G(D) - S^{n-1}$, we obtain the aforementioned inequality.

Step 2: Any non-trivial linear combination

$$h := \sum_{i=1}^{n} b_i \left(\pi_i \circ G\right) : D \longrightarrow \mathbb{R}$$

attains its maximum only at the point $F^{-1}(x_0) \in S^{n-1}$.
Not difficult!

Let us see that Theorem 2 recovers the 2-lineability of $\widehat{C}(\mathbb{R})$.

For $a, b \in \mathbb{R}^2$, by (a, b) we mean the open line segment in $\mathbb{R}^2$ from a to b. Take $Y = [0, +\infty) \subset \mathbb{R}$, a continuous bijection $F\colon [0, +\infty) \longrightarrow S^1$, a homeomorphism $g\colon (-\infty, 0) \longrightarrow (F(0), (0, 0)) \subset \mathbb{R}^2$ such that $\lim_{x \to 0} g(x) = F(0)$.

Now, it is enough to define

$$G\colon \mathbb{R} \longrightarrow \mathbb{R}^2 \ , \ G(x) = \begin{cases} F(x), \ x \geq 0, \\ g(x), \ x < 0. \end{cases}$$

Let us see that Theorem 2 recovers the 2-lineability of $\widehat{C}(\mathbb{R})$. For $a, b \in \mathbb{R}^2$, by (a, b) we mean the open line segment in $\mathbb{R}^2$ from a to b. Take $Y = [0, +\infty) \subset \mathbb{R}$, a continuous bijection $F\colon [0, +\infty) \longrightarrow S^1$, a homeomorphism $g\colon (-\infty, 0) \longrightarrow (F(0), (0, 0)) \subset \mathbb{R}^2$ such that $\lim_{x \to 0} g(x) = F(0)$.

Now, it is enough to define

$$G\colon \mathbb{R} \longrightarrow \mathbb{R}^2 \;,\; G(x) = \begin{cases} F(x), & x \geq 0, \\ g(x), & x < 0. \end{cases}$$

Let us see that Theorem 2 recovers the 2-lineability of $\widehat{C}(\mathbb{R})$. For $a, b \in \mathbb{R}^2$, by (a, b) we mean the open line segment in $\mathbb{R}^2$ from a to b. Take $Y = [0, +\infty) \subset \mathbb{R}$, a continuous bijection $F\colon [0, +\infty) \longrightarrow S^1$, a homeomorphism $g\colon (-\infty, 0) \longrightarrow (F(0), (0, 0)) \subset \mathbb{R}^2$ such that $\lim\limits_{x \to 0} g(x) = F(0)$. Now, it is enough to define

$$G\colon \mathbb{R} \longrightarrow \mathbb{R}^2\ ,\ G(x) = \begin{cases} F(x), & x \geq 0, \\ g(x), & x < 0. \end{cases}$$

Let us see that Theorem 2 recovers the 2-lineability of $\widehat{C}(\mathbb{R})$. For $a, b \in \mathbb{R}^2$, by (a, b) we mean the open line segment in $\mathbb{R}^2$ from a to b. Take $Y = [0, +\infty) \subset \mathbb{R}$, a continuous bijection $F\colon [0, +\infty) \longrightarrow S^1$, a homeomorphism $g\colon (-\infty, 0) \longrightarrow (F(0), (0, 0)) \subset \mathbb{R}^2$ such that $\lim_{x \to 0} g(x) = F(0)$. Now, it is enough to define

$$G\colon \mathbb{R} \longrightarrow \mathbb{R}^2 \ , \ G(x) = \begin{cases} F(x), & x \geq 0, \\ g(x), & x < 0. \end{cases}$$

Let us see that Theorem 2 recovers the 2-lineability of $\widehat{C}(\mathbb{R})$. For $a, b \in \mathbb{R}^2$, by (a, b) we mean the open line segment in $\mathbb{R}^2$ from a to b. Take $Y = [0, +\infty) \subset \mathbb{R}$, a continuous bijection $F\colon [0, +\infty) \longrightarrow S^1$, a homeomorphism $g\colon (-\infty, 0) \longrightarrow (F(0), (0, 0)) \subset \mathbb{R}^2$ such that $\lim_{x\to 0} g(x) = F(0)$.

Now, it is enough to define

$$G\colon \mathbb{R} \longrightarrow \mathbb{R}^2 \ , \ G(x) = \begin{cases} F(x), & x \geq 0, \\ g(x), & x < 0. \end{cases}$$

Let us see that Theorem 2 recovers the 2-lineability of $\widehat{C}(\mathbb{R})$.
For $a, b \in \mathbb{R}^2$, by (a, b) we mean the open line segment in $\mathbb{R}^2$ from a to b. Take $Y = [0, +\infty) \subset \mathbb{R}$, a continuous bijection $F\colon [0, +\infty) \longrightarrow S^1$, a homeomorphism $g\colon (-\infty, 0) \longrightarrow (F(0), (0, 0)) \subset \mathbb{R}^2$ such that $\lim_{x \to 0} g(x) = F(0)$.
Now, it is enough to define

$$G\colon \mathbb{R} \longrightarrow \mathbb{R}^2\ ,\ G(x) = \begin{cases} F(x), & x \geq 0, \\ g(x), & x < 0. \end{cases}$$

Generalizing (C)

We prove that, on the one hand, $\widehat{C}(S^{m-1}) \cup \{0\}$ contains an m-dimensional subspace of $C(S^{m-1})$; and on the other hand, for every compact subset K of $\mathbb{R}^m$, $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$.

More precisely:

Theorem

Let $n \geq 2$ and $m \geq 1$ be positive integers. Then $m < n$ if, and only if, for every compact set $K \subset \mathbb{R}^m$, there is no n-dimensional subspace of $C(K)$ contained in $\widehat{C}(K) \cup \{0\}$.

Generalizing (C)

We prove that, on the one hand, $\widehat{C}(S^{m-1}) \cup \{0\}$ contains an m-dimensional subspace of $C(S^{m-1})$; and on the other hand, for every compact subset K of $\mathbb{R}^m$, $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$.

More precisely:

Theorem

Let $n \geq 2$ and $m \geq 1$ be positive integers. Then $m < n$ if, and only if, for every compact set $K \subset \mathbb{R}^m$, there is no n-dimensional subspace of $C(K)$ contained in $\widehat{C}(K) \cup \{0\}$.

Generalizing (C)

We prove that, on the one hand, $\widehat{C}(S^{m-1}) \cup \{0\}$ contains an m-dimensional subspace of $C(S^{m-1})$; and on the other hand, for every compact subset K of $\mathbb{R}^m$, $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$.
More precisely:

Theorem

Let $n \geq 2$ and $m \geq 1$ be positive integers. Then $m < n$ if, and only if, for every compact set $K \subset \mathbb{R}^m$, there is no n-dimensional subspace of $C(K)$ contained in $\widehat{C}(K) \cup \{0\}$.

Generalizing (C)

We prove that, on the one hand, $\widehat{C}(S^{m-1}) \cup \{0\}$ contains an m-dimensional subspace of $C(S^{m-1})$; and on the other hand, for every compact subset K of $\mathbb{R}^m$, $\widehat{C}(K) \cup \{0\}$ does not contain an $(m+1)$-dimensional subspace of $C(K)$.
More precisely:

Theorem

Let $n \geq 2$ and $m \geq 1$ be positive integers. Then $m < n$ if, and only if, for every compact set $K \subset \mathbb{R}^m$, there is no n-dimensional subspace of $C(K)$ contained in $\widehat{C}(K) \cup \{0\}$.

We omit the proof of this result, since it is very technical. We need to prove 4 technical lemmas + introduce a special function + hardwork + Borsuk-Ulam Theorem to obtain the result.

We omit the proof of this result, since it is very technical. We need to prove 4 technical lemmas + introduce a special function + hardwork + Borsuk-Ulam Theorem to obtain the result.

We omit the proof of this result, since it is very technical. We need to prove 4 technical lemmas + introduce a special function + hardwork + Borsuk-Ulam Theorem to obtain the result.

We omit the proof of this result, since it is very technical. We need to prove 4 technical lemmas + introduce a special function + hardwork + Borsuk-Ulam Theorem to obtain the result.

We omit the proof of this result, since it is very technical. We need to prove 4 technical lemmas + introduce a special function + hardwork + Borsuk-Ulam Theorem to obtain the result.

We omit the proof of this result, since it is very technical. We need to prove 4 technical lemmas + introduce a special function + hardwork + Borsuk-Ulam Theorem to obtain the result.

We omit the proof of this result, since it is very technical. We need to prove 4 technical lemmas + introduce a special function + hardwork + Borsuk-Ulam Theorem to obtain the result.

An infinite dimensional example

Let K be a compact subset of $\mathbb{R}^m$. We saw that, for $m < n$, there is no n-dimensional subspace of $C(K)$ formed, up to the origin, by functions that attain the maximum only at one point. Now we show that if we allow K to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(K)$ may contain, up to the origin, an infinite dimensional subspace of $C(K)$.

Let K be the following subset of ℓ_2:

$$K = \left\{ \left(\frac{a_n}{n} \right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that K is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n} \right]$. Since the Hilbert cube is compact, to prove that K is compact it is enough to show that it is closed (it is not difficult to prove this!).

An infinite dimensional example

Let K be a compact subset of $\mathbb{R}^m$. We saw that, for $m < n$, there is no n-dimensional subspace of $C(K)$ formed, up to the origin, by functions that attain the maximum only at one point. Now we show that if we allow K to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(K)$ may contain, up to the origin, an infinite dimensional subspace of $C(K)$.

Let K be the following subset of ℓ_2:

$$K = \left\{ \left(\frac{a_n}{n} \right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that K is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$. Since the Hilbert cube is compact, to prove that K is compact it is enough to show that it is closed (it is not difficult to prove this!).

An infinite dimensional example

Let K be a compact subset of $\mathbb{R}^m$. We saw that, for $m < n$, there is no n-dimensional subspace of $C(K)$ formed, up to the origin, by functions that attain the maximum only at one point. Now we show that if we allow K to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(K)$ may contain, up to the origin, an infinite dimensional subspace of $C(K)$.

Let K be the following subset of ℓ_2:

$$K = \left\{ \left(\frac{a_n}{n} \right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that K is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n} \right]$. Since the Hilbert cube is compact, to prove that K is compact it is enough to show that it is closed (it is not difficult to prove this!).

An infinite dimensional example

Let K be a compact subset of $\mathbb{R}^m$. We saw that, for $m < n$, there is no n-dimensional subspace of $C(K)$ formed, up to the origin, by functions that attain the maximum only at one point. Now we show that if we allow K to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(K)$ may contain, up to the origin, an infinite dimensional subspace of $C(K)$.

Let K be the following subset of ℓ_2:

$$K = \left\{ \left(\frac{a_n}{n} \right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that K is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n} \right]$. Since the Hilbert cube is compact, to prove that K is compact it is enough to show that it is closed (it is not difficult to prove this!).

An infinite dimensional example

Let K be a compact subset of $\mathbb{R}^m$. We saw that, for $m < n$, there is no n-dimensional subspace of $C(K)$ formed, up to the origin, by functions that attain the maximum only at one point. Now we show that if we allow K to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(K)$ may contain, up to the origin, an infinite dimensional subspace of $C(K)$.

Let K be the following subset of ℓ_2:

$$K = \left\{ \left(\frac{a_n}{n} \right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that K is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$. Since the Hilbert cube is compact, to prove that K is compact it is enough to show that it is closed (it is not difficult to prove this!).

An infinite dimensional example

Let K be a compact subset of $\mathbb{R}^m$. We saw that, for $m < n$, there is no n-dimensional subspace of $C(K)$ formed, up to the origin, by functions that attain the maximum only at one point. Now we show that if we allow K to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(K)$ may contain, up to the origin, an infinite dimensional subspace of $C(K)$.

Let K be the following subset of ℓ_2:

$$K = \left\{ \left(\frac{a_n}{n} \right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that K is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$. Since the Hilbert cube is compact, to prove that K is compact it is enough to show that it is closed (it is not difficult to prove this!).

An infinite dimensional example

Let K be a compact subset of $\mathbb{R}^m$. We saw that, for $m < n$, there is no n-dimensional subspace of $C(K)$ formed, up to the origin, by functions that attain the maximum only at one point. Now we show that if we allow K to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(K)$ may contain, up to the origin, an infinite dimensional subspace of $C(K)$.

Let K be the following subset of ℓ_2:

$$K = \left\{ \left(\frac{a_n}{n} \right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that K is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$. Since the Hilbert cube is compact, to prove that K is compact it is enough to show that it is closed (it is not difficult to prove this!).

An infinite dimensional example

Let K be a compact subset of $\mathbb{R}^m$. We saw that, for $m < n$, there is no n-dimensional subspace of $C(K)$ formed, up to the origin, by functions that attain the maximum only at one point. Now we show that if we allow K to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(K)$ may contain, up to the origin, an infinite dimensional subspace of $C(K)$.

Let K be the following subset of ℓ_2:

$$K = \left\{ \left(\frac{a_n}{n}\right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that K is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]$. Since the Hilbert cube is compact, to prove that K is compact it is enough to show that it is closed (it is not difficult to prove this!).

Now we proceed to show that $\widehat{C}(K) \cup \{0\}$ contains an infinite dimensional subspace of $C(K)$. Consider the function

$$F \colon K \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n}\right)_{n=1}^{\infty}\right) = (a_n)_{n=1}^{\infty} \ .$$

By $\pi_j \colon \ell_2 \longrightarrow \mathbb{R}$ we mean the projection onto the j-th coordinate, $j \in \mathbb{N}$. For each j, the function

$$\pi_j \circ F \colon K \longrightarrow \mathbb{R}$$

is continuous because $\pi_j \circ F = j \cdot \pi_j$. It is clear that the functions $\pi_j \circ F, j \in \mathbb{N}$, are linearly independent. Let $f := \sum_{j=1}^{k} b_j(\pi_j \circ F)$ be a non-trivial linear combination of these continuous functions. Writing $b = (b_1, \ldots, b_k, 0, 0, \ldots) \in \ell_2$ we have $f(x) = \langle b, F(x) \rangle$ for every $x \in K$. As $b \in \ell_2^k$ and $\|F(x)\|_2 \leq 1$ for every $x \in K$,

$$f(x) = \langle b, F(x) \rangle < \left\langle b, \frac{b}{\|b\|_2} \right\rangle$$

whenever $F(x) \neq \frac{b}{\|b\|_2}$.

Now we proceed to show that $\widehat{C}(K) \cup \{0\}$ contains an infinite dimensional subspace of $C(K)$. Consider the function

$$F\colon K \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n}\right)_{n=1}^{\infty}\right) = (a_n)_{n=1}^{\infty} \, .$$

By $\pi_j\colon \ell_2 \longrightarrow \mathbb{R}$ we mean the projection onto the j-th coordinate, $j \in \mathbb{N}$. For each j, the function

$$\pi_j \circ F\colon K \longrightarrow \mathbb{R}$$

is continuous because $\pi_j \circ F = j \cdot \pi_j$. It is clear that the functions $\pi_j \circ F, j \in \mathbb{N}$, are linearly independent. Let $f := \sum_{j=1}^{k} b_j(\pi_j \circ F)$ be a non-trivial linear combination of these continuous functions. Writing $b = (b_1, \ldots, b_k, 0, 0, \ldots) \in \ell_2$ we have $f(x) = \langle b, F(x) \rangle$ for every $x \in K$. As $b \in \ell_2^k$ and $\|F(x)\|_2 \leq 1$ for every $x \in K$,

$$f(x) = \langle b, F(x) \rangle < \left\langle b, \frac{b}{\|b\|_2} \right\rangle$$

whenever $F(x) \neq \frac{b}{\|b\|_2}$.

Now we proceed to show that $\widehat{C}(K) \cup \{0\}$ contains an infinite dimensional subspace of $C(K)$. Consider the function

$$F\colon K \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n}\right)_{n=1}^{\infty}\right) = (a_n)_{n=1}^{\infty} \, .$$

By $\pi_j\colon \ell_2 \longrightarrow \mathbb{R}$ we mean the projection onto the j-th coordinate, $j \in \mathbb{N}$. For each j, the function

$$\pi_j \circ F\colon K \longrightarrow \mathbb{R}$$

is continuous because $\pi_j \circ F = j \cdot \pi_j$. It is clear that the functions $\pi_j \circ F, j \in \mathbb{N}$, are linearly independent. Let $f := \sum_{j=1}^{k} b_j(\pi_j \circ F)$ be a non-trivial linear combination of these continuous functions. Writing $b = (b_1, \ldots, b_k, 0, 0, \ldots) \in \ell_2$ we have $f(x) = \langle b, F(x) \rangle$ for every $x \in K$. As $b \in \ell_2^k$ and $\|F(x)\|_2 \leq 1$ for every $x \in K$,

$$f(x) = \langle b, F(x) \rangle < \left\langle b, \frac{b}{\|b\|_2} \right\rangle$$

whenever $F(x) \neq \frac{b}{\|b\|_2}$.

Now we proceed to show that $\widehat{C}(K) \cup \{0\}$ contains an infinite dimensional subspace of $C(K)$. Consider the function

$$F\colon K \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n}\right)_{n=1}^{\infty}\right) = (a_n)_{n=1}^{\infty} \, .$$

By $\pi_j\colon \ell_2 \longrightarrow \mathbb{R}$ we mean the projection onto the j-th coordinate, $j \in \mathbb{N}$. For each j, the function

$$\pi_j \circ F\colon K \longrightarrow \mathbb{R}$$

is continuous because $\pi_j \circ F = j \cdot \pi_j$. It is clear that the functions $\pi_j \circ F, j \in \mathbb{N}$, are linearly independent. Let $f := \sum_{j=1}^{k} b_j(\pi_j \circ F)$ be a non-trivial linear combination of these continuous functions. Writing $b = (b_1, \ldots, b_k, 0, 0, \ldots) \in \ell_2$ we have $f(x) = \langle b, F(x)\rangle$ for every $x \in K$. As $b \in \ell_2^k$ and $\|F(x)\|_2 \leq 1$ for every $x \in K$,

$$f(x) = \langle b, F(x)\rangle < \left\langle b, \frac{b}{\|b\|_2}\right\rangle$$

whenever $F(x) \neq \frac{b}{\|b\|_2}$.

Now we proceed to show that $\widehat{C}(K) \cup \{0\}$ contains an infinite dimensional subspace of $C(K)$. Consider the function

$$F\colon K \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n}\right)_{n=1}^{\infty}\right) = (a_n)_{n=1}^{\infty} .$$

By $\pi_j\colon \ell_2 \longrightarrow \mathbb{R}$ we mean the projection onto the j-th coordinate, $j \in \mathbb{N}$. For each j, the function

$$\pi_j \circ F\colon K \longrightarrow \mathbb{R}$$

is continuous because $\pi_j \circ F = j \cdot \pi_j$. It is clear that the functions $\pi_j \circ F, j \in \mathbb{N}$, are linearly independent. Let $f := \sum_{j=1}^{k} b_j(\pi_j \circ F)$ be a non-trivial linear combination of these continuous functions. Writing $b = (b_1, \ldots, b_k, 0, 0, \ldots) \in \ell_2$ we have $f(x) = \langle b, F(x)\rangle$ for every $x \in K$. As $b \in \ell_2^k$ and $\|F(x)\|_2 \leq 1$ for every $x \in K$,

$$f(x) = \langle b, F(x)\rangle < \left\langle b, \frac{b}{\|b\|_2}\right\rangle$$

whenever $F(x) \neq \frac{b}{\|b\|_2}$.

Now we proceed to show that $\widehat{C}(K) \cup \{0\}$ contains an infinite dimensional subspace of $C(K)$. Consider the function

$$F\colon K \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n}\right)_{n=1}^{\infty}\right) = (a_n)_{n=1}^{\infty} \, .$$

By $\pi_j \colon \ell_2 \longrightarrow \mathbb{R}$ we mean the projection onto the j-th coordinate, $j \in \mathbb{N}$. For each j, the function

$$\pi_j \circ F \colon K \longrightarrow \mathbb{R}$$

is continuous because $\pi_j \circ F = j \cdot \pi_j$. It is clear that the functions $\pi_j \circ F, j \in \mathbb{N}$, are linearly independent. Let $f := \sum_{j=1}^{k} b_j(\pi_j \circ F)$ be a non-trivial linear combination of these continuous functions. Writing $b = (b_1, \ldots, b_k, 0, 0, \ldots) \in \ell_2$ we have $f(x) = \langle b, F(x) \rangle$ for every $x \in K$. As $b \in \ell_2^k$ and $\|F(x)\|_2 \leq 1$ for every $x \in K$,

$$f(x) = \langle b, F(x) \rangle < \left\langle b, \frac{b}{\|b\|_2} \right\rangle$$

whenever $F(x) \neq \frac{b}{\|b\|_2}$.

Now we proceed to show that $\widehat{C}(K) \cup \{0\}$ contains an infinite dimensional subspace of $C(K)$. Consider the function

$$F\colon K \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n}\right)_{n=1}^{\infty}\right) = (a_n)_{n=1}^{\infty} \, .$$

By $\pi_j \colon \ell_2 \longrightarrow \mathbb{R}$ we mean the projection onto the j-th coordinate, $j \in \mathbb{N}$. For each j, the function

$$\pi_j \circ F \colon K \longrightarrow \mathbb{R}$$

is continuous because $\pi_j \circ F = j \cdot \pi_j$. It is clear that the functions $\pi_j \circ F, j \in \mathbb{N}$, are linearly independent. Let $f := \sum_{j=1}^{k} b_j(\pi_j \circ F)$ be a non-trivial linear combination of these continuous functions. Writing $b = (b_1, \ldots, b_k, 0, 0, \ldots) \in \ell_2$ we have $f(x) = \langle b, F(x) \rangle$ for every $x \in K$. As $b \in \ell_2^k$ and $\|F(x)\|_2 \leq 1$ for every $x \in K$,

$$f(x) = \langle b, F(x) \rangle < \left\langle b, \frac{b}{\|b\|_2} \right\rangle$$

whenever $F(x) \neq \frac{b}{\|b\|_2}$.

Now we proceed to show that $\widehat{C}(K) \cup \{0\}$ contains an infinite dimensional subspace of $C(K)$. Consider the function

$$F\colon K \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n}\right)_{n=1}^{\infty}\right) = (a_n)_{n=1}^{\infty} \, .$$

By $\pi_j\colon \ell_2 \longrightarrow \mathbb{R}$ we mean the projection onto the j-th coordinate, $j \in \mathbb{N}$. For each j, the function

$$\pi_j \circ F\colon K \longrightarrow \mathbb{R}$$

is continuous because $\pi_j \circ F = j \cdot \pi_j$. It is clear that the functions $\pi_j \circ F, j \in \mathbb{N}$, are linearly independent. Let $f := \sum_{j=1}^{k} b_j(\pi_j \circ F)$ be a non-trivial linear combination of these continuous functions. Writing $b = (b_1, \ldots, b_k, 0, 0, \ldots) \in \ell_2$ we have $f(x) = \langle b, F(x) \rangle$ for every $x \in K$. As $b \in \ell_2^k$ and $\|F(x)\|_2 \leq 1$ for every $x \in K$,

$$f(x) = \langle b, F(x) \rangle < \left\langle b, \frac{b}{\|b\|_2} \right\rangle$$

whenever $F(x) \neq \frac{b}{\|b\|_2}$.

As F is a bijection onto the closed unit ball of ℓ_2, there is a unique $y \in K$ such that $F(y) = \frac{b}{\|b\|_2}$. This shows that f attains its maximum at y. An adaptation of the argument used in the proof of Theorem 2 guarantees that this maximum is unique.

As F is a bijection onto the closed unit ball of ℓ_2, there is a unique $y \in K$ such that $F(y) = \frac{b}{\|b\|_2}$. This shows that f attains its maximum at y. An adaptation of the argument used in the proof of Theorem 2 guarantees that this maximum is unique.

[1] V. I. Gurariy, L. Quarta,
On lineability of sets of continuous functions,
J. Math. Anal. App. **294** (2004) 62–72.

Thank You Very Much!!!