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Regulated Solutions and Periodicity for Nonlinear Integral Equations on Time Scales in Banach Spaces

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Introduction

Here we are concerned with the integral

$$\int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s \alpha(s) x(s)$$

in the context of discontinous functions x with values in a general Banach space, in a time scale $\mathbb{T}$ (that is, $\mathbb{T}$ is a closed non void subset of the real numbers $\mathbb{R}$, and $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$ in $\mathbb{R}$).

Discontinuous functions arise in a natural way when we are describing phenomenon in Mathematics, Physics or Technology , as for instance in collision theory when the displacement changes suddenly the direction, or even as in electronics where the components of an electrical field are discontinuous across the surface of an electric conductor and across the surface that separates two different dielectrics. More recent is the considerations on discontinuous functions that appears when using the classical play operator with variable characteristics in the theory on hysteresis n*M. Brokate, P. Krejčí, Duality in the space of regulated functions and the play operator, Mathematische Zeitschrift 245, 667–688, (2003).*

For a general time scale $\mathbb{T}$ (see the definition below) some examples of Banach spaces containing discontinuous functions are:

1 - the spaces $L_p([\mathrm{a},\mathrm{b}]_{\mathbb{T}}, X)$, $1 \leq p < \infty$ of the measurable functions in the Bochner or Lebesgue sense on the time scale $\mathbb{T} = \mathbb{R}$, or the space $BV([\mathrm{a},\mathrm{b}]_{\mathbb{T}}, X)$ of functions of bounded variation, the Sobolev spaces , etc.

Regulated Functions

2 - the space of the regulated functions on time scales $\mathbb{T}$, $f \epsilon G([a,b]_{\mathbb{T}}, \mathrm{X})$

We say that $f:[a,b]_{\mathbb{T}} \longrightarrow \mathrm{X}$ is *regulated* if $\mathrm{f}(\mathrm{t}^{+})$ [respectively $\mathrm{f}(\mathrm{t}^{-})$] is defined for all points t not isolated at right [respectively at left] in $\mathbb{T}$. If moreover f is left-continuous at these points then we write $f \epsilon G^{-}([a,b]_{\mathbb{T}}, \mathrm{X})$.

Time scales

Settled in 1988 by Stefan Hilger

S. Hilger, Analysis on measure chains - a unified approach to continuous and discrete calculus, Results Math. 18,18-56, (1990).

The Calculus on time scales was created with the purpose in to unify the theory of continuous time and the discrete dynamical systems .

The Riemann integral on time scales is the subject of several works in the literature

M. Bohner, A. Peterson, Dynamic Equations on Time Scales: an introduction with applications , Birkäuser 368 p. (2001).

G. Sh. Guseinov, Integration on time scales, J. Math. Anal. Appl. 285 ,107–127, (2003)

R. Agarwal, M.Bohner, D. O'Regan, A.Peterson, Dynamic equations on time scales: a survey, Journal of Computational and Applied Mathematics 141 ,1–26, (2002).

Here we take the notations in Guseinov.

Considerations on the integral theory (in the Riemann or in the generalized Riemann senses) on time scales includes, among others, the Riemann delta and nabla-integral, alfa-integral, the Lebesgue and nabla-integrals, and the Henstock-Perron-Kurzweil ones

It is more recent (2009) the work by Mozyrska-Pawleszwicz-Torres

D. Mozyrska, E. Pawluszewicz, D. F. M. Torres, Riemann-Stieltjes integral in time scales , The Australian Journal of Mathematical Analysis and Applications 7(1), 1-14, (2010).

considering the Riemann-Stieltjes integral.

With the purpose, among others, in to overcome difficulties when trying to extend class of the functions considered in the Riemann-Stieltjes integral to the discontinuous one ,we will be extending the concept of integral to the Cauchy one. It will be doing in the coming section.

A generalized Riemann-Stieltjes integral on time scales $\mathbb{T}$

Here we introduce the right Cauchy-Stieltjes integral on time scales. The considerations on this type of integral when working in the time scales environment are appropriated for several reasons that become more and more evident with its use.

The Riemann-Stieltjes integral do not possess, in the general, the additive rule:

$$\int_a^c f(t) \,\square\, g(t) + \int_c^b f(t) \,\square\, g(t) = \int_a^b f(t) \,\square\, g(t).$$

When using the Cauchy-Stieltjes integral we will be retrieving these fundamental properties.

Moreover by considering the right Cauchy-Stieltjes integral, see T. H. Hildebrandt, we have, roughly speaking, a great degree of compatibility among the integral, the left continuous functions in time scale $\mathbb{T}$, and the kind of filtering convergence in the definition of the integral itself.

Consider the class $\wp_{[a,b]_{\mathbb{T}}}$ of all partitions of $\mathbb{I} = [a,b]_{\mathbb{T}}$.

Let us take $f \in \mathcal{F}([a,b]_{\mathbb{T}}, X)$ and $\alpha : [a,b]_{\mathbb{T}} \to \mathrm{L(X,W)}$. The (Cauchy) sum associated to

$$P = \{a = t_0, t_1, \dots, t_n = b\} \in \wp_{[a,b]_{\mathbb{T}}}$$

of f relatively to α, is defined by:

$$\sigma_P(\mathrm{f};\alpha) = \sum_{\mathrm{i}=0}^{\mathrm{n}-1}[\alpha(\mathrm{t_{i+1}}) - \alpha(\mathrm{t_i})].\mathrm{f}(\mathrm{t_{i+1}})$$

The (right)-Cauchy-Stieltjes integral $(r - CS)$ of f relatively to α,

$$(rC - S)\int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s).f(s)$$

or simply $\int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s).f(s)$ - is the value

$$\int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s).f(s) = \lim_{P \in \wp_{[a,b]_{\mathbb{T}}}} \sigma_P(\mathrm{f};\alpha)$$, since it exists.

Recall that the expression $\lim_{P \in \wp_{[a,b]_{\mathbb{T}}}} \sigma_P(\mathrm{f};\alpha) = z$ in a general topological space

means: for every neighborhood V of z, there exists $P_0 \in \wp_{[a,b]_{\mathbb{T}}}$ with $\sigma_P(\mathrm{f};\alpha) \in V$ whenever $P \in \wp_{[a,b]_{\mathbb{T}}}$ and $P \supseteq P_0$.

Existence of the (rC-S) integral on time scales

To proceed let us be specifying the concepts.

Definition (semivariation):

Let be $\wp_{[a,b]_{\mathbb{T}}}$ the set of the partitions of $[a,b]_{\mathbb{T}}$ and $P \in \wp_{[a,b]_{\mathbb{T}}}$ and $\alpha: [a,b]_{\mathbb{T}} \longrightarrow L(X,W)$. The function α is of bounded semivariation and we write $\alpha \epsilon \mathrm{SV}([a,b]_{\mathbb{T}}, L(X,W))$ if

$$SV[\alpha] = \sup_{P \in \wp_{[a,b]_{\mathbb{T}}}} SV_P[\alpha] < \infty$$

with

$$SV_P[\alpha] = \sup\{\| \sum_{i=1}^{|P|} [\alpha(t_i) - \alpha(t_{i-1})] x_i \|_W ; x_i \epsilon X; \|x_i\| < 1\}$$

Definition (bounded variation):

If $\alpha: [a, b] \to Z$, the variation of α in $[a, b]_{\mathbb{T}}$ is

$$BV[\alpha] = \sup_{P \in \wp_{[a,b]_{\mathbb{T}}}} \{\sum_{i=1}^{|P|} ||\alpha(t_i) - \alpha(t_{i-1})||$$

If $BV[\alpha] < \infty$, we say that α is of bounded variation and we write $\alpha \in BV([a, b]_{\mathbb{T}}, Z)$

Proposition:

i - $BV([a, b]_{\mathbb{T}}, Z) \subset G([a, b]_{\mathbb{T}}, Z)$

ii - $BV([a, b]_{\mathbb{T}}, L(X, W) \subset SV([a, b]_{\mathbb{T}}, Z)$, and $SV[\alpha] < BV[\alpha]$, and the equality holds if and only if $\dim W < \infty$.

Observe that $SV[\alpha]$ is a semi-norm. It is a norm on the class of all $\alpha \epsilon$ SV$([a,b]_{\mathbb{T}}, L(X))$ with $\alpha(a) = 0$.If α is such that satisfies these conditions we write $\alpha \epsilon \mathrm{SV}_0([a,b]_{\mathbb{T}}, L(X))$.

In the following we recall some conditions on α and f for the existence of (at right) Cauchy-Stieltjes integral at right $(rC - S) \int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s). f(s)$ as well as some of its first properties .

Theorem (Damasceno & Barbanti,Th.II.1.(b)) : For every finite $\mathbb{T}$ let be $f \epsilon G^{-}([a,b]_{\mathbb{T}}, X)$ and $\alpha \, \epsilon$ SV$([a,b]_{\mathbb{T}}, L(X,W))$ .Then $\mathcal{I}_{\alpha}(f) = \int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s). f(s) \in W$ there exists, and $\mathcal{I}_{\alpha} \in L(G \quad ([a,b], X), W)$. Furthermore $\|\mathcal{I}_{\alpha}\| \leq SV[\alpha]$, and for every $c \in [a,b]_{\mathbb{T}}$, we have:

$$\int_{[a,c]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s). f(s) + \int_{[c,b]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s). f(s) = \int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s). f(s) .$$

THE OPERATORS TO BE REPRESENTED BY MEANS OF AN INTEGRAL

(1) $A \in \; L(G^{-}([a,b]_{\mathbb{T}}, \mathrm{X}), Y)$

(2) $A \in \; L(G^{-}([a,b]_{\mathbb{T}}, \mathrm{X}), G([a,b]_{\mathbb{T}}, \mathrm{X}))$

(3) $A \in \; L(G^{-}([a,b]_{\mathbb{T}}, \mathrm{X}), G([a,b]_{\mathbb{T}}, \mathrm{X}))$ **A causal**

L. Barbanti, M. Federson, B.C. Damasceno and G.N. Silva "*Volterra-Stieltjes integral equations on regulated functions in Banach spaces in time scales*", Springer Proc. in Math.and Statistics, 47, 2013 (to appear).

Definition (causal operator). If $\mathcal{F}([a,b]_{\mathbb{T}}, X)$ is a function space we say that $A \in L(\mathcal{F}([a,b]_{\mathbb{T}}, X), \mathcal{F}([a,b]_{\mathbb{T}}, X))$ is causal (or not anticipative) if *for every* $x \epsilon\, \mathcal{F}([a,b]_{\mathbb{T}}, X)$ *and for all* $c \epsilon [a,b]_{\mathbb{T}}$, the following holds $for\ every\ \tau \epsilon [a,c]_{\mathbb{T}}$, $x(\tau) = 0 \implies Ax(\tau) = 0$.

Representing the operator A in (1)

The kernel α**:**

$$\alpha \in SV_0([a,b]_{\mathbb{T}}, L(X,W)) \to \mathcal{I}_\alpha \in L(G^-([a,b],X),W)$$

Is an isometry $\|\mathcal{I}_\alpha\| = SV[\alpha]$ **and**

$$A(f) = \mathcal{I}_\alpha(f) = \int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s\, \alpha(s).f(s) \in W$$

where for every t **and every** $x \in X$ **we have** $\alpha(s)x = A(\chi_{(a,t]}x)$

Consequence: when $W = \mathbb{R}$

$$(G^-([a,b]_{\mathbb{T}})' \approx SV([a,b]_{\mathbb{T}}, X')$$

NON-LINEAR VOLTERRA-STIELTJES INTEGRAL EQUATIONS

$$(K) \qquad x(t) + \int_{[a,b]_{\mathbb{T}}} \mathbb{D}_s \, \mathrm{K}(t,s) f\big(s, x(s)\big) = u(t)$$

- $x, u \in G([a,b]_{\mathbb{T}}, X)$
- $f: [a,b]_{\mathbb{T}} \times G([a,b]_{\mathbb{T}}, X) \to X$
- $K \in G_{\Delta}^{\sigma} SV^{u}([a,b]_{\mathbb{T}}, L(X)) \qquad \| \quad K\| = SV^{u}[k]$

$F: G^{-}([a,b]_{\mathbb{T}}, X) \to G^{-}([a,b]_{\mathbb{T}}, X)$
$F(x)(t) = f\big(t, x(t)\big)$ the Nemiitskji operator associated to f

Lemma: F is well defined iff f is regulated left continuous in the first variable and Lipschitzian in the second one.

Existence of regulated solutions for (K)

Let be (K), and F well defined.Let be L the Lipschitz constant for f and

$$L < \frac{1}{SV^{u}[K]}$$

Then (K) has a unique solution in $G([a,b]_{\mathbb{T}}, X)$

Linear Case

- Linear case [when $f\big(t, x(t)\big) = x(t)$]:
- If $SV^u[K] < 1$ then the linear equation (K) has one and only one regulated solution

Returning Solutions in (K)

Suppose all the conditions on (K) satisfied.

We say that $x_u \in G([a,b]_{\mathbb{T}}, X)$, is a *returning solution* for (K) -forced by u- *in time* t_1, t_2 if there exists t_1 and t_2 $(t_1 \neq t_2)$ in $[a,b]_{\mathbb{T}}$ such that $x_u(t_1) = x_u(t_2)$.

<u>Proposition</u>: Let be $t_1 \neq t_2$ in $[a,b]_{\mathbb{T}}$ and $K(t_1, s) = K(t_2, s)$ for every $s \in [a,b]_{\mathbb{T}}$.

A necessary and sufficient condition for the existence of a *returning solution in time* t_1, t_2, is the existence of a forcing u such that $u(t_1) = u(t_2)$

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