Some results related with the Riemann-Lebesgue lemma II

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Introduction

If $f:\mathbb{R}\to\mathbb{R}$ is a Lebesgue integrable function, $f\in L(\mathbb{R})$, its Fourier transform is defined as

$$\widehat{f}(s)=\int_{-\infty}^{\infty} f(t)e^{-ist}dt. \tag{1}$$

If f is not in $L(\mathbb{R})$ its Fourier Transform may not exist.

- The Fourier transform of $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\dfrac{sinx}{x} & x\neq 0\\ 1 & x=0,\end{cases}$$

there not exists for $s=1$.

Introduction

The Riemann-Lebesgue lemma in $L(\mathbb{R})$

If $f \in L(\mathbb{R})$, then:

- $\widehat{f}$ is continuous on $\mathbb{R}$
- $\lim_{|s|\to\infty} \widehat{f}(s) = 0$.

Introduction

If f is not in $L(\mathbb{R})$, the Riemann-Lebesgue lemma may be not valided

Example

$g(x) = \exp(ix^2)$

$$\widehat{g}(s) = \sqrt{\pi}\exp(i(\pi - s^2)/4).$$

$\widehat{g}(s)$ no tend to zero when s tend to infinity.

[E. Talvila, *Henstock-Kurzweil Fourier transforms*, Illinois J. Math., **46** (2002), 1207-1226.]

Notation

- $L(I)$: Lebesgue integrable functions
- $L^2(\mathbb{R})$: Quadratic Lebesgue integrable functions
- $HK(I)$: Henstock-Kurzweil integrable functions
- $BV(I)$: Bounded variation functions
- $BV_0(I)$: Bounded variation functions that vanish at infinity

$$I = [a, b], [a, \infty), (-\infty, b], or \mathbb{R}$$

$BV_0(\mathbb{R})$ space

- $f : [a, \infty) \to \mathbb{R}$ is of bounded variation if exist $M > 0$ such that, for all $b \geq a$,

$$V(f, [a, b]) \leq M.$$

- $$V(f, [a, \infty)) = \sup_{a \leq b} V(f, [a, b]).$$

- $BV_0(\mathbb{R})$=$\{ f \in BV(\mathbb{R}) : lim_{t \to \pm\infty} f(t) = 0\}$.

A subspace of $BV_0(\mathbb{R})$ space

Proposition 1. Suppose $f : [a, \infty) \to \mathbb{R}$ is not identically zero, it is continuous and periodic with period $b - a$. Let $\alpha,\ \beta$ positives such that $\alpha + \beta > 1$ with $\beta \leq 1$. Then $f_{\alpha,\beta} : [a^{1/\alpha}, \infty) \to \mathbb{R}$ defined by

$$f_{\alpha,\beta}(t) = \frac{f(t^\alpha)}{t^\beta} \tag{2}$$

is in $HK([a^{1/\alpha}, \infty)) \setminus L^1([a^{1/\alpha}, \infty))$.

[F. J. Mendoza T., M. G. Morales M., et al., *Several aspects around the Riemann-Lebesgue lemma*, j. Adv. Res. PureMath., 5 (3), 2013, 33-48.]

Proposition 2. Let $\beta > \alpha > 0$ be fixed with $\beta + \alpha > 1$. Suppose $f : [a, \infty) \to \mathbb{R}$ is a bounded and continuous function, with bounded derivative. Then $f_{\alpha,\beta} : [a^{1/\alpha}, \infty) \to \mathbb{R}$ defined by

$$f_{\alpha,\beta}(t) = \frac{f(t^\alpha)}{t^\beta}$$

belongs to $BV([a^{1/\alpha}, \infty))$.

[F. J. Mendoza T., M. G. Morales M., et al., *Several aspects around the Riemann-Lebesgue lemma*, j. Adv. Res. PureMath., 5 (3), 2013, 33-48.]

Corollary 3. Let $a,\ \alpha,\ \beta$ be such that $0 < \alpha < \beta \leq 1$ and $\beta + \alpha > 1$. Suppose that $f : [a, \infty) \to \mathbb{R}$ satisfies both the hypotheses of Propositions 1 and 2. Then

$$f_{\alpha,\beta} \in HK([a^{1/\alpha}, \infty)) \cap BV([a^{1/\alpha}, \infty)) \setminus L^1([a^{1/\alpha}, \infty)). \qquad (3)$$

[F. J. Mendoza T., M. G. Morales M., et al., *Several aspects around the Riemann-Lebesgue lemma*, j. Adv. Res. PureMath., 5 (3), 2013, 33-48.]

Corollary 4. Let $a,\ \alpha,\ \beta$ be such that $0 < \alpha < \beta \leq 1$ and $\beta + \alpha > 1$, and let $h \in BV([-a^{1/\alpha},\ a^{1/\alpha}])$. Suppose that the function $f : [a, \infty) \to \mathbb{R}$ satisfies both the hypotheses of Propositios 1 and 2. Then $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} h(t) & \text{if } \ t \in (-a^{1/\alpha},\ a^{1/\alpha}), \\ \dfrac{f(|t|^{\alpha})}{|t|^{\beta}} & \text{if } \ f \in (-\infty,\ -a^{1/\alpha}] \cup [a^{1/\alpha},\ \infty) \end{cases}$$

is in $HK(\mathbb{R}) \cap BV(\mathbb{R}) \setminus L(\mathbb{R})$.

[F. J. Mendoza T., M. G. Morales M., et al., *Several aspects around the Riemann-Lebesgue lemma*, j. Adv. Res. Pure Math., 5 (3), 2013, 33-48.]

Example

$$sin_\beta^\alpha : \mathbb{R} \to \mathbb{R}; \quad sin_\beta^\alpha(t) = \chi_{[\pi^{1/\alpha},\infty)}(t)\frac{sin(t^\alpha)}{t^\beta},$$
$$\cos_\beta^\alpha : \mathbb{R} \to \mathbb{R}; \quad \cos_\beta^\alpha(t) = \chi_{[(\pi/2)^{1/\alpha},\infty)}(t)\frac{\cos(t^\alpha)}{t^\beta}.$$

Where $\alpha,\ \beta$ are such that $0 < \alpha < \beta \leq 1$ and $\beta + \alpha > 1$.

[F. J. Mendoza T., M. G. Morales M., et al., *Several aspects around the Riemann-Lebesgue lemma*, j. Adv. Res. Pure Math., 5 (3), 2013, 33-48.]

Others inclusion relations of $BV_0(\mathbb{R})$

Proposition 5. $L(\mathbb{R}) \not\subseteq HK(\mathbb{R}) \cap BV(\mathbb{R})$

Proposition 6.$(*)$ $HK(\mathbb{R}) \cap BV(\mathbb{R}) \subset BV_0(\mathbb{R})$

Proposition 7. $HK(\mathbb{R}) \cap BV(\mathbb{R})$ is dense in $L^2(\mathbb{R})$

$(*)$ [S. Sánchez-Perales, F. J. Mendoza T. and J. A. Escamilla R., H*enstock-Kurzweil integral transforms*, International Journal of Mathematics and Mathematical Sciences, (2012), 11 pages, 2012.]

A Generalized Riemann-Lebesgue lemma in BV_0

Let $\varphi \in HK_{loc}(\mathbb{R})$ such that $\Phi(t) = \int_0^t \varphi(x)dx$ is bounded on $\mathbb{R}$. If $f \in BV_0(\mathbb{R})$, then

- $H(w) = \int_{-\infty}^{\infty} f(t)\varphi(wt)dt$ is defined on $\mathbb{R} \setminus \{0\}$,
- it is continuous on $\mathbb{R} \setminus \{0\}$
- $$\lim_{|w|\to\infty} H(w) = 0.$$

[F. J. Mendoza T., M. G. Morales M., et al., *Several aspects around the Riemann-Lebesgue lemma*, j. Adv. Res. PureMath., 5 (3), 2013, 33-48.]

proof

For $w \in \mathbb{R}$, we define $\varphi_w(t) = \varphi(wt)$. Since $\varphi \in HK_{loc}(\mathbb{R})$, then φ and φ_w are in $HK([0,b])$, for $b > 0$. By Jordan decomposition, f can be represented as the difference of f_1 and f_2 which are nondecreasing functions belonging to $BV_0(\mathbb{R})$. Therefore, by Chartier-Dirichlet Test, $f\varphi_w \in HK([0,\infty])$. Moreover, by the Multiplier Theorem we have, for $w \neq 0$,

$$\begin{aligned} \int_0^\infty f(t)\varphi(wt)dt &= -\int_0^\infty \frac{\Phi(wt)}{w} df(t) \\ &= -\int_0^\infty \frac{\Phi(wt)}{w} df_1(t) \\ &\quad + \int_0^\infty \frac{\Phi(wt)}{w} df_2(t), \end{aligned} \tag{4}$$

where $df_i(t)$ is the Lebesgue-Sieltjes measure generated by f_i,

proof

Let $\beta > 0$ be and let M the upper bound of $|\Phi|$. For each $w \in [\beta, \infty)$ we have that

$$\left| \frac{\Phi(wt)}{w} \right| \leq \frac{M}{\beta}. \tag{5}$$

Because of $\Phi(wt)/w$ is continuous over $[\beta, \infty)$ and the measures $df_i(t)$ are finite, then by the Dominated Convergence Theorem applied to right side integrals in (4), it follows that

$$\lim_{w \to w_0} H(w) = H(w_0),$$

for each $w_0 \in [\beta, \infty)$. Since β is arbitrary, we obtain the continuity of H on $(0, \infty)$.

proof

In addition, by (4), we have for $w \in (0, \infty)$ that

$$\left| \int_0^\infty f(t)\varphi(wt)dt \right| \leq \frac{M}{|w|} Var(f; [0, \infty]).$$

Therefore we conclude that

$$\lim_{|w| \to \infty} \int_0^\infty f(t)\varphi(wt)dt = 0.$$

Similar arguments are valid for the interval $(-\infty, 0]$, which yields to complete the proof.

Corollary

The Riemann-Lebesgue lemma in BV_0

If $f \in BV_0(\mathbb{R})$, then the Fourier transform $\widehat{f}(s)$ exists for all $s \in \mathbb{R} \setminus \{0\}$, and has the following properties

(i) $\widehat{f} : \mathbb{R} \to \mathbb{C}$ *is continuous at* $\mathbb{R} \setminus \{0\}$.

(ii) $\lim\limits_{|s|\to\infty} \widehat{f}(s) = 0$.

[F. J. Mendoza T, *On pointwise inversion of the Fourier transform of BVo functions*, Annals of Functional Analysis 2 (2010), 2010, 112-120.]

A Pointwise Inversion Fourier Theorem

The Dirichlet-Jordan theorem in BV_0

If $f \in BV_0(\mathbb{R})$, then, for each $x \in \mathbb{R}$,

$$\lim_{M\to\infty, \varepsilon\to 0} \frac{1}{2\pi} \int\limits_{\varepsilon<|s|<M} e^{ixs}\widehat{f}(s)ds = \frac{1}{2}\{f(x+0)+f(x-0)\}. \quad (6)$$

[F. J. Mendoza T, *On pointwise inversion of the Fourier transform of BVo functions*, Annals of Functional Analysis 2 (2010), 2010, 112-120.]

Some consequences

- Because of $BV_0(\mathbb{R})$ does not have inclusion relations with the Lebesgue space. The Riemann-Lebesgue lemma is not exclusive for Lebesgue integrable functions.
- The classical Dirichlet-Jordan theorem in $L(\mathbb{R})$ is a particular case of the Dirichlet-Jordan theorem in BV_0. This is a pointwise inversion Fourier theorem.

Some consequences

- There exist functions in $L^2(\mathbb{R}) \setminus L(\mathbb{R})$ such that their Henstock-Fourier transforms exist as in (1) and, for each $x \in \mathbb{R}$, the expression (6) is true.
- A version of the Plancherel theorem is possible from a dense set in $L^2(\mathbb{R})$ which does not have inclusion relation with $L(\mathbb{R})$.

¡Obrigado!