Universality in the Laguerre-Pólya class

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A question of Pólya

- **Pólya, 1926:** ‘What properties of the the function $K(t)$ are sufficient to secure that the integral

$$\phi(z) = \int_{-\infty}^{\infty} K(t) e^{itz} dt$$

  has only real zeros?
  The origin of this artificial question is the Riemann hypothesis concerning the zeta-function.”
- The Laguerre-Pólya class $\mathcal{LP}$ consists of real entire functions of order at most two, with only real zeros. It is the complement, in the sense of local uniform convergence, of polynomials, whose zeros are only real. The latter polynomials are usually called hyperbolic ones.
- The kernels (even kernels) $K(t)$ for which the corresponding Fourier transform (cosine transform) is in $\mathcal{LP}$ are usually called $\mathcal{LP}-$kernels.

The Riemann hypothesis

- Let $\zeta(z)$ be the Riemann zeta-function. Define the Riemann ξ function:

$$\xi(iz) = \frac{1}{2}(z^2 - 1/4)\pi^{-z/2-1/4}\Gamma(z/2 + 1/4)\zeta(z + 1/2),$$

where Γ is the Gamma function. Then the Riemann hypothesis states that ξ, represented also as the Fourier transform

$$\xi(z/2) = 4\int_{-\infty}^{\infty} \Phi(t)e^{izt}dt,$$

$$\Phi(t) = \sum_{n=1}^{\infty}(2n^4\pi^2e^{9t} - 3n^2\pi e^{5t})\exp(-n^2\pi e^{4t}),$$

has only real zeros.

The Lee-Yang theorems and Lee-Yang measures

- **Lee and Yang, 1952:** Studied the Ising model with Hamiltonian

$$H = -\sum J_{jk}\sigma_j\sigma_k - \sum z_j\sigma_j,$$

which is ferromagnetic one if all $J_{jk} \geq 0$ and z_j correspond to an external magnetic field.

- The partition function is given by

$$Z = \int e^{-H} d\mu(\sigma_1)\cdots d\mu(\sigma_N),$$

where the measure $d\mu$ is a symmetric (even) measure on $\mathbb{R}$, for which the integrals exist.

- They observed that a phase transition in a system depends on the location of the zeros of Z, considered as a function of z_j (the space where z_j "live" is callsed the fugacity space).

The Lee-Yang theorems

- **Lee and Yang, 1952, 1953:** In two papers they proved that when the measure corresponds to the simple Ising model, i.e., when the measure has only two equal masses (jumps) at ± 1, then the zeros of $Z(z, \ldots, z)$ are purely imaginary. More precisely, they proved that $Z(z_1, \ldots, z_N) \neq 0$ if $\Re(z_k) > 0$. This helped them to describe the phase transition of the Ising model. For this result they were awarded the Nobel Prize.
- **B. Simon and Griffits, 1973:** Extended the result to continuous distributions. They used an approximation of those distributions by superpositions of Ising models.
- **Charles Newman, 1974:** Established a more general result saying that the Lee-Yang theorem holds for certain model provided it holds for a zero action.
- **Lieb e Sokal, 1981:** Generalized Newman's result from one-dimensional measures to measures in high dimensional Euclidean spaces.

The Lee-Yang measures

- A rapidly decreasing measure on the real line is said to possess the Lee-Yang property, or simply to be a Lee-Yang measure if its Fourier transform defines an entire function of order at most two whose zeros are all real.
- The essence of the above mentioned results is that if a model in Statistical Mechanics is related to a Lee-Yang measure, then the Lee-Yang theorem would hold for the corresponding partition function. In this case one might be able to characterize the phase transition in the corresponding model.
- Thus, it is of extreme importance to characterize the Lee-Yang measures.

Equivalence

- It is clear that Polya's problem and the problem about characterization of the Lee-Yang measures are equivalent!
- Both areas have had some "contact"and it is described in a comments by Marc Kac on a paper of Pólya, published in Volume 2 of Pólya's Collected Papers.
- It turns out that Lee and Yang knew some of Pólya's results and somehow extended one of his results to establish their celebrated theorem.
- However, it seems that only very recently this interplay has began to be explored by experts on zeros of entire functions and on Statistical Mechanics.

Two general results

- **de Bruijn, 1949**: If $\varphi(z)$ is any function in $\mathcal{LP}$ order at most one, then there is a unique C^{∞} function $K(t)$, such that

$$e^{-z^2/2}\varphi(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2/2}K(t)e^{itz}\,dt.$$

- **de Bruijn, 1950:** Let $h(t)$ be an entire function such that its derivative is the uniform limit, on compact subsets of the complex plane, of a sequence of polynomials, all of whose zeros lie on the imaginary axis. If $h(t)$ is non-constant with $h(t) = h(-t)$, and if $h(t) \geq 0$ for $t \in \mathbb{R}$, then the entire function

$$\int_{-\infty}^{\infty} \exp(-h(t))e^{izt}\,dt$$

has only real zeros.

A new characterization; A class of polynomials

- Let $d\mu(t)$ be an symmetric (even) probability Lebesgue measure on the real line, such that the sequence of polynomials $p_n(z)$, orthogonal with respect to $d\mu$, is uniquely defined.
- Define also

$$P_n(z_1, z_2, \cdots, z_n) = \frac{\begin{vmatrix} p_1(z_1) & p_2(z_1) & \cdots & \cdots & p_n(z_1) \\ p_1(z_2) & p_2(z_2) & \cdots & \cdots & p_n(z_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_1(z_n) & p_2(z_n) & \cdots & \cdots & p_n(z_n) \end{vmatrix}}{\begin{vmatrix} 1 & z_1 & \cdots & \cdots & z_1^{n-1} \\ 1 & z_2 & \cdots & \cdots & z_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & \cdots & \cdots & z_n^{n-1} \end{vmatrix}}.$$

A characterization of the Lee-Yang measures

- **DKD:**The Fourier transform of $d\mu$,

$$\phi(z) = \int_{-\infty}^{\infty} e^{izt} d\mu(t)$$

possesses only real zeros if and only if $P_n(z_1, \ldots, z_n) \neq 0$ for $\Re(z_k) > 0$, for every $n \in \mathbb{N}$.

- This is equivalent to the fact that, for every $n \in \mathbb{N}$, the polynomial $P_n(z, z, \cdots, z)$, which coincides with the Wronskian of $p_1(z), \cdots, p_n(z)$, vanishes only when z is a purely imaginary number, that is, $P_n(z, z, \cdots, z) = 0$ only when $z = iy$, $y \in \mathbb{R}$.

Important facts

- One can prove that the above polynomials, with certain rescaling, converge locally uniformly to the Fourier thansform.
- The numerator polynomials in the definition of $P_n(z_1, \ldots, z_n)$, coincide with the so-called are Slater determinants. In the theory of Hilbert spaces Slater's determinants play an importan role as a basis in the so-called Fock spaces which are tensor products of n copies of a Hilbert space. In Quantum Mechanics antisymmetric (fermion) and symmetric (boson) Fock spaces are considered. Slater determinants (permanents) are exactly the corresponding wave functions, so that they describe states in a system of n particles. Since electrons are fermions, Slater's detrminants are wave functions of n electrons.

More facts

- The partition functions describe completely the system. The energy, entropy, preasure, Helmholtz free energy can be obtained from the partition function.
- Consider a grand partition function as a Laplace transform. The fugacity variable (external potential) z can be viewed also as the so-called chemical potential, sometimes denoted by μ. The above result means that this partition function is a limit of specific wave functions (Slater determinants). Observe that the lattrer descibe exactly the interaction of electrons, so thay are related to the chemical properties of the particles.

An equivalent formulation

- Let $K(t)$ be any even positive weight function whose moments

$$\mu_j = \int_{-a}^{a} t^j K(t) dt, \quad j = 0, 1, \dots,$$

exist, and let $p_n(x)$, $n = 0, 1, \dots$ be the polynomials, orthogonal with respect to $K(t)$ in $(-a, a)$. Let the Fourier transform

$$\varphi(z) = \int_{-a}^{a} K(t) e^{izt} dt,$$

of $K(t)$ be an entire function of order at most two. Then $\varphi(z)$ belongs to the Laguerre-Pólya class if and only if, for any positive even integer n, all the zeros of the Wronskian

$$W(p_1(z), p_2(x), \dots, p_n(z))$$

are purely imaginary.

Another result

- If $K(t)$ obeys the above requirements, then φ belongs to the Laguerre-Pólya class if and only if

$$W(p_2(ix), p_3(ix), \ldots, p_{2n}(ix)) \geq 0, \quad x \in \mathbb{R}$$

for every $n \in \mathbb{N}$.

Back to entire functions: two conjectures of Pólya

- **Pólya, 1943, Conjecture 1:** If the order of the real entire function f is less than 2, and f has only a finite number of non-real zeros, then its derivatives, from a certain one onwards, will have no non-real zeros at all.
- **Conjecture 2:** If the order of the real entire function f is greater than 2, and f has only a finite number of nonreal zeros, then the number of non-real zeros of $f^{(n)}$ tends to infinity as $n \to \infty$.
- **Craven, Csordas and Smith, 1987:** Proved Conjecture 1
- **Ki and Kim, 1990, 2000:** Proved that the statement of Conjecture 1 holds if $f = P\,\varphi$, where P is a real polynomial and $\varphi \in \mathcal{LP}$.
- **Bergweiler and Eremenko, 2006:** Proved Conjecture B. Actually, they proved that its conclusion holds if $f = P\,\varphi$, with a real polynomial P and a real entire function φ with real zeros that does not belong to $\mathcal{LP}$.

Further differentiation; almost equal spacing?

- **Ki 2006:** There exist sequences A_n and C_n, with $C_n \to 0$, such that

$$\lim_{n\to\infty} A_n \,\xi^{(2n)}(C_n z) = \cos z$$

uniformly on compact subsets of $\mathbb{C}$.

- **D. W. Farmer and R. C. Rhoades, 2005:** Conjectured the above result inspider by the following result they proved: Suppose f is a real entire function of order 1 which has only real zeros, and $n_+(r) \sim n_-(r) \sim \kappa\, r$. Then there exist sequences A_n, B_n, and D_n with D_n bounded, such that

$$\lim_{n\to\infty} A_n \exp(B_n z)\, f^{(n)}(z/\kappa + D_n) = \cos(\pi z)$$

uniformly compact subsets of $\mathbb{C}$.

Differentiation and spacing; integration

- **Ki 2006:** Let $a > 0$ and $b \in \mathbb{R}$. Let $K(t)$ be a continuous function on the real line, such that $K(t) = K(-t)$ and

$$K(t) = \exp(-ae^t)\exp(bt)(1 + o(1)) \;\; \text{as} \;\; t \to \infty.$$

  Let

$$f(z) = \int_{-\infty}^{\infty} K(t)\, e^{izt}\, dt.$$

  Then there exist sequences A_n and C_n, with $C_n \to 0$, such that

$$\lim_{n\to\infty} \; A_n\, f^{(2n)}(C_n\, z) = \cos z$$

  uniformly on compact subsets of $\mathbb{C}$.

- **Wintner, 1947:** Proved that

$$\xi^{(-1)}(z) = \int_{-\infty}^{\infty} \Phi(t)\sin(zt)/t\, dt > 0 \;\; \text{for} \;\; z > 0.$$

Reisz and Stoyanoff's result

- Let f be a function with only real zeros denoted by x_j. Let $m(f) = inf|x_j - x_i|$ where the infimum is taken over all $j \neq i$.
- **Marcel Riesz, never published:** Let $f(x)$ be a hyperbolic polynomial. If a is real, then $m(f' + af) \geq m(f)$, with strict inequality provided the zeros of f are simple.
- **Stoyanoff 1926:** Published a proof, attributing the resulf to M. Riesz.
- **Farmer and Rhoades, 2005:** Entended the result to functions $f(x)$ belonging to $\mathcal{LP}1$.
- If p is a hyperbolic polynomial of degree n, let $\tilde{p}$ be the polynomial of degree $n-1$ with zeros $(x_i + x_{i+1})/2$. Riesz-Stoyanoff's result implies $m(p') \geq m(p)$. It is obvious that $m(\tilde{p}) \geq m(p)$. Farmer and Rhoades conjectured that $m(p') \geq m(\tilde{p})$.
- **DKD and Vladimir Kostov, 2010:** Gave counterexamples both for hyperbolic polynomials and functions from $\mathcal{LP}1$.

Reisz-Stoyanoff and Ki for $\mathcal{LP}$?

- Reisz-Stoyanoff's result and Farmer-Rhoades' generalization state that the spacing of zeros become less dense under differentiation in $\mathcal{LP}1$.
- Ki's result states that the zeros tend to be equaly spaced for a subclass of $\mathcal{LP}1$ (conditional under the RH).
- What about the whole class $\mathcal{LP}$?
- There is a simple counterexample where none of these phenomena occurs!
- Let

$$\varphi(z) = \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}.$$

The inequalities $T_k := \gamma_k^2 - \gamma_{k-1}\gamma k + 1 \geq 0$ are called the Turán inequalities and are necessary conditions for $\varphi \in \mathcal{LP}$.

The universality conjecture.

- **DKD and Vladimir Kostov, 2011:** Found a new set of necessary conditions. It turned out that quantities similar to T_k/γ_k^2 showed a similar asymptotic behavious for a large class of functions in $\mathcal{LP}$.
- **Conjecture (universality):** Quantities like these posses similar asymptotic behaviour for all $\varphi \in \mathcal{LP}$.
- This is in the spirit of the universality in RMT. There pair correlation statistics of eigenvalues of some random matrices exibit the same behaviour. Montgomery-Dyson!

Another connection: The BMV conjecture or Stahl's theorem

- **Bessis, Moussa, Villani, 1976:** Studying certain perturbations of "partition functions" of system in Statistical Mechanics, tried to use Padé approximations. Their attempts led them to formulate the following conjecture:
- Let A and B be Hermitian $n \times n$ matrices and B be positive semi-definite. Then the function

$$f(t) := Tr \exp(A - tB), \quad t \geq 0,$$

can be represented as a Laplace trasfrom,

$$f(t) = \int e^{-ts} d\mu_{A,B}(s)$$

of a positive measure $\mu_{A,B}$ in $\mathbb{R}_+ = [0, \infty)$.

An equivalent form of BMV and Stahl's proof

- A characterization of Bernstein yields that the latter is equivalent to the fact that $f(t)$ is completely monotone, that is,

$$(-1)^n f^{(n)}(t) \geq 0 \ \text{ for all } \ n \in \mathbb{N} \ \text{ and } t \in \mathbb{R}_+.$$

- **Herbert Stahl, 2011:** posted a 56 page paper on arXiv with a proof of BMV conjecture. The third version is much shorter, 27 pages.
- The paper will appear in Annals of Mathematics.
- **Eremenko, 2013:** Wrote a nice survey on Stahl's approach.

An open question concerning the above results

- Let us consider the measures $\mu_{A,B}(s)$ which obey the above properties, extend them as even ones to the entire real line and call the resulting measures Stahl's measures.
- Consider

$$\phi(z) = f(iz) + f(-iz) = 2\int_0^\infty \cos zs \; d\mu_{A,B}(s)$$

  Then

$$\phi(z) = \int_{-\infty}^\infty \exp(izs) \; d\mu_{A,B}(s)$$

- Then we ask: Are there any Stahl measures that are also Lee-Yang measures?
- This suggests the problem to characterize the pairs of matrices A and B which generate such measures. Do such measures, if they exist, describe interesting models in Statistical Mechanics?

Constantino Tsallis' model

- Constantino Tsallis, who was born in Grece and grew up in Argentina, is a naturalized Brazilian physicist working at CBPF in Rio de Janeiro.
- Tsallis is credited with introducing the notion of what is known as Tsallis entropy and Tsallis statistics in his 1988 paper "Possible generalization of Boltzmann-Gibbs statistics" published in the Journal of Statistical Physics. The generalization is considered to be one of the most viable and applicable candidates for formulating a theory of non-extensive thermodynamics. The resulting theory is not intended to replace Boltzmann-Gibbs statistics, but rather supplement it, such as in the case of anomalous systems characterized by non-ergodicity or metastable states.

Possible q-analogs?

- From mathematical point of view, Tsallis' model is defined by a partition function where the exponent in the partition function in the Boltzmann-Gibbs model

$$Z = \int e^{-H} d\mu(\sigma_1) \cdots d\mu(\sigma_N),$$

is substituted by the q-exponential function.

- What about the phase transition in the Tsallis model?
- What should be the role of the corresponding Lee-Yang measures? Do the q-orthogonal polynomials play a similar role in the characterization of those measures?