Nonlinear operators in the spaces of functions of bounded variation in the sense of Jordan

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Amplitude of forced vibration

Definition

One can prove that the nonlinear integral equation

$$x(t) = \omega^2 \int_0^1 G(t,s)\rho(s)x(s)\mathrm{d}s + \int_0^1 G(t,s)q(s)\mathrm{d}s, \qquad t \in [0,1],$$

where $G(t,s) = \begin{cases} t(1-s), & \text{for } 0 \le t \le s \le 1, \\ s(1-t), & \text{for } 0 \le s \le t \le 1, \end{cases}$

under suitable assumptions on functions ρ and q, and the constant ω, possesses a unique continuous solution on $[0,1]$, being a function of bounded variation in the sense of Jordan.

Space $BV[0,1]$

Definition

Let $x\colon [0,1] \to \mathbb{R}$. The number

$$\mathrm{var}(x) = \sup \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|,$$

where the supremum is taken over all the partitions $0 = t_0 < \ldots < t_n = 1$ of the interval $[0,1]$ is said to be the variation of the function x in the sense of Jordan over the interval $[0,1]$.

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Remark

The space

$$BV[0,1] = \left\{ x\colon [0,1] \to \mathbb{R} : \operatorname{var}(x) < +\infty \right\}$$

endowed with the norm $\|x\|_{BV} = |x(0)| + \operatorname{var}(x)$ is a Banach space.

Convolution operators

Definition

If f and g are real-valued functions defined on $\mathbb{R}$, then their convolution is defined by

$$f \star g(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt,$$

provided the above integral exists.

Convolution operators

Theorem (Talvila, 2002)

Let $f \in HK$ and $g \in BV$. Then $f \star g$ exists on $\mathbb{R}$ and

$$|f \star g(x)| \leq \|f\|(\inf |g| + \mathrm{var}(g)) \quad \textit{for all } x \in \mathbb{R},$$

where $\|f\|$ denotes the Alexiewicz norm of the function f.

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Theorem (Talvila, 2002)

Let $f \in HK$ and $g \in L^1 \cap BV$. Then $f \star g$ exists on $\mathbb{R}$ and

$$\|f \star g\| \leq \|f\| \|g\|_1.$$

Superposition operators

Definition

For given function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ we define the nonautonomous superposition operator F, generated by f, as

$$F(x)(t) = f(t, x(t)),$$

where x is a real-valued function defined on $[0,1]$.

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Definition

In the case when $f : \mathbb{R} \to \mathbb{R}$, the operator F, generated by f, is said to be the autonomous superposition operator.

Superposition operators

Theorem (Josephy, 1981)

For $f : \mathbb{R} \to \mathbb{R}$ *the superposition* $f \circ x$ *belongs to the space* $BV[0,1]$ *for all* $x \in BV[0,1]$ *if and only if* f *satisfies the local Lipschitz condition on* $[0,1]$.

Superposition operators

Theorem (Josephy, 1981)

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Theorem (Ljamin, 1986)

Suppose that the function $f(s, \cdot)$ satisfies the Lipschitz condition on $\mathbb{R}$, uniformly in $s \in [0,1]$, and the function $f(\cdot, u)$ is of bounded variation in the sense of Jordan on $[0,1]$, uniformly in $u \in \mathbb{R}$. Then the nonautonomous superposition operator F, generated by f, maps the space $BV[0,1]$ into itself and it is bounded.

Counterexample of Ljamin's theorem

Example (Maćkowiak, 2012)

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 0, & \forall n \in \{2,3,\ldots\} : x \neq c_n \text{ or } y \notin I_n, \\ \frac{1}{n}\left(1 - \frac{|y-c_n|}{w_n}\right), & \exists n \in \{2,3,\ldots\} : x = c_n \text{ and } y \in I_n, \end{cases}$$

where $c_n = 1 - \frac{1}{n}$, $w_n = \frac{1}{2n}$, $I_n = (c_n - w_n, c_n + w_n)$, for $n = 2, 3, \ldots$.
For any $x \in [0,1]$, the function $f(x,\cdot)$ satisfies the Lipschitz condition uniformly in x, with a Lipschitz constant not greater that 2 and $\mathrm{var}(f(\cdot,y)) \leq 22$ for any $y \in \mathbb{R}$.

Counterexample of Ljamin's theorem

Example (continuation)

However, the nonautonomous superposition operator generated by f does not map the space $BV[0,1]$ into itself. Indeed, let $u(x) = x$ and $g(x) = f(x, u(x))$ for $x \in [0,1]$. Obviously, $\mathrm{var}(u) = 1$. Moreover,

$$g(x) = \begin{cases} \frac{1}{n}, & \text{if } \; x = c_n, \\ 0, & \text{if } \; x \neq c_n, \end{cases}$$

what gives $\mathrm{var}(g) = +\infty$.

Sufficient condition

Theorem (D. Bugajewska, 2010)

Suppose that a function $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$, $(t,u)\to f(t,u)$ *satisfies the local Lipschitz condition on* $\mathbb{R}$, *uniformly in* $t\in[0,1]$. *Moreover, assume that for every* $r>0$ *there exists a constant* $M_r>0$ *such that for every* $k\in\mathbb{N}$, *every partition* $t_0<\ldots<t_k$ *of* $[0,1]$ *and every* $u_0,\ldots,u_{k-1}\in[-r,r]$, *the following implication holds*

$$\sum_{i=1}^{k-1}|u_i-u_{i-1}|\le r \Longrightarrow \sum_{i=1}^{k}|f(t_i,u_{i-1})-f(t_{i-1},u_{i-1})|<M_r. \quad (1)$$

Then the superposition operator F, *generated by* f, *maps the space* $BV[0,1]$ *into itself and it is locally bounded.*

Nonlinear integral operators

Theorem (D. Bugajewski, 2003)

Let $I = [0,1]$. Assume that:

(a) $g : I \to \mathbb{R}$ *is a BV-function;*

(b) $f : \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitz function;*

(c) $K : I \times I \to \mathbb{R}$ *is a function such that* $\mathrm{var}(K(\cdot, s)) \leq M(s)$ *for a.e.* $s \in I$, *where* $M : I \to \mathbb{R}_+$ *is integrable in the Lebesgue sense and* $K(t, \cdot)$ *is integrable in the Lebesgue sense for every* $t \in I$.

Then there exists a number $\rho > 0$ *such that for every* λ *satisfying* $|\lambda| < \rho$, *the equation*

$$x(t) = g(t) + \lambda \int_I K(t,s) f(x(s)) ds, \quad t \in I, \ \lambda \in \mathbb{R}$$

possesses a unique BV-solution, defined on I.

Nonlinear integral operators

Theorem (D. Bugajewski, 2003)

Suppose that (a) *and* (b) *are satisfied. Assume also that*

(d) $T = \{(t,s): \ 0 \le t \le a, \ 0 \le s \le t\}$ *and* $K : T \to \mathbb{R}$ *is a function such that* $|K(s,s)| + \mathrm{var}(K(\cdot,s);[s,a]) \le m(s)$ *for a.e.* $s \in I$*, where* $m : I \to \mathbb{R}_+$ *is integrable in the Lebesgue sense and* $K(t,\cdot)$ *is integrable in the Lebesgue sense on* $[0,t]$ *for every* $t \in I$*.*

Then there exists an interval $J \subset I$ *such that the equation*

$$x(t) = g(t) + \int_0^t K(t,s) f(x(s)) ds \ \ t \in I$$

possesses a unique BV*-solution, defined on* J*.*

Appendix

Theorem (D.B., D.B., P.K., P.M., 2013)

Suppose that a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $(t,u) \to f(t,u)$ *satisfies the local Lipschitz condition on* $\mathbb{R}$, *uniformly in* $t \in [0,1]$ *and that the superposition operator* F, *generated by* f, *maps the space* $BV[0,1]$ *into itself and it is locally bounded. Then for arbitrary positive number* r *there exists a constant* $M_r > 0$ *such that for every* $k \in \mathbb{N}$, *every partition* $t_0 < \ldots < t_k$ *of* $[0,1]$ *and every* $u_0, \ldots, u_{k-1} \in [-r, r]$ *the implication* (1) *holds.*

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