Extremal problems and Hilbert spaces of entire functions

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Extremal problems

- Goal of approximation theory: Approximate [1] a certain object [2] by an object in a certain class [3] in such a way to minimize a certain quantity [4].

Here we are going to talk about five different (one-sided) extremal problems:

1. Given $f : \mathbb{R} \to \mathbb{R}$, find the entire function $K : \mathbb{C} \to \mathbb{C}$ of exponential type at most $\tau > 0$ that minimizes

$$\int_{-\infty}^{\infty} |K(x) - f(x)| \, \mathrm{d}\mu(x).$$

Note: $\tau(F) := \limsup_{|z| \to \infty} |z|^{-1} \, |\log K(z)|$.

- Bernstein, Krein, Akhiezer, Nagy, Beurling (1930's), Selberg (1950's).

Extremal problems (cont.)

2. Given $f : [-1,1] \to \mathbb{R}$, find the polynomial K of degree at most d that minimizes

$$\int_{-1}^{1} |K(x) - f(x)| \, \mathrm{d}\mu(x).$$

   - Freud, Jackson, Szegö, Geronimus.

3. Given $f : \mathbb{R} \to \mathbb{R}$ periodic of period 1, find the trigonometric polynomial K of degree at most d that minimizes

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |K(x) - f(x)| \, \mathrm{d}\mu(x).$$

Extremal problems (cont.)

4. Given $f : \mathbb{R}^N \to \mathbb{R}$, find the entire function $K : \mathbb{C}^N \to \mathbb{C}$ of exponential type at most $\tau > 0$ that minimizes

$$\int_{\mathbb{R}^N} |K(\mathbf{x}) - f(\mathbf{x})| \, \mathrm{d}\mu(\mathbf{x}).$$

Note: $\tau(F) := \limsup_{\|\mathbf{z}\|_K \to \infty} \|\mathbf{z}\|_K^{-1} \, |\log K(\mathbf{z})|$, where

$\|\mathbf{z}\|_K = \sup\{|\mathbf{z} \cdot \mathbf{x}|; \, \mathbf{x} \in K\}$,

K is a compact convex symmetric set on $\mathbb{R}^N$.

- Vaaler ('96)

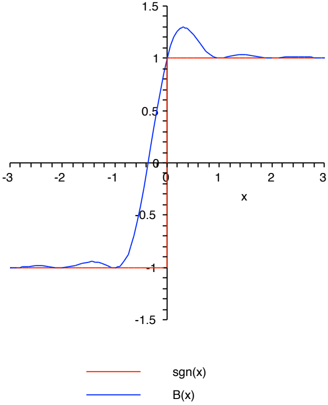
5. Given $f : S^{N-1} \to \mathbb{R}$ find polynomial $K : \mathbb{R}^N \to \mathbb{R}$ of degree at most d that minimizes

$$\int_{S^{N-1}} |K(\mathbf{x}) - f(\mathbf{x})| \, \mathrm{d}\mu(\mathbf{x}).$$

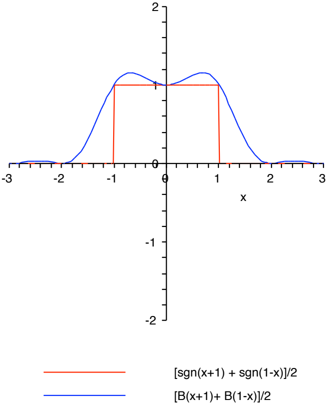
- Li-Vaaler ('98)

Two classical pictures of problem (1)

Beurling 1930's



Selberg 1950's



$$B(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{\mathrm{sgn}_+(n)}{(z-n)^2} + \frac{2}{z} \right\}$$

$$\int_{-\infty}^{\infty} \{B(x) - \mathrm{sgn}(x)\}\, \mathrm{d}x = 1$$

$$\chi_{[a,b]}(x) = \tfrac{1}{2}\{\mathrm{sgn}(x-a) + \mathrm{sgn}(b-x)\}$$

$$M_{[a,b]}(x) = \tfrac{1}{2}\{B(x-a) + B(b-x)\}$$

Some applications (1) - Hilbert-type inequalities

- (Hilbert's inequality) Let $\{a_m\}_{m=1}^{M}$ be a sequence of complex numbers. Then
$$\left|\sum_{\substack{m,n=1\\m\neq n}}^{M} \frac{a_m\,\overline{a_n}}{m-n}\right| \leq \pi \sum_{m=1}^{M} |a_j|^2.$$
  - Proved by Hilbert (with $C = 2\pi$) and later by Schur (with $C = \pi$ sharp).
  - Montgomery and Vaughan ('76) replace the denominator by a sequence $\{\delta_m\}$ of well-spaces real numbers. (i.e. $|\delta_m - \delta_n| \geq 1$.)
- *N*-dimensional version: Let $\{a_m\}_{m=1}^{M}$ be complex numbers and $\{\delta_m\}_{m=1}^{M}$ be well-spaces vectors in $\mathbb{R}^N$ (i.e. $|\delta_m - \delta_n| \geq 1$.).
$$-C_1(\alpha)\sum_{m=1}^{M}|a_j|^2 \leq \sum_{\substack{m,n=1\\m\neq n}}^{M} \frac{a_m\,\overline{a_n}}{|\delta_m - \delta_n|^{\alpha}} \leq C_2(\alpha)\sum_{m=1}^{M}|a_j|^2,$$
where $-N-1 \leq \alpha < 0$ (for the majorant $-N-1 < \alpha < -N$).

Some applications (2) - Erdös-Turán inequalities

- Given a sequence $\{x_i\}_{i=1}^M$ of real numbers modulo 1, we can define the discrepancy of this sequence by

$$\Delta(\{x_i\}) = \sup_{0\leq y<1} \left\{ \frac{1}{M} \left| \sum_{m=1}^{M} \psi(x_i - y) \right| \right\},$$

where $\psi(x) = (x - \lfloor x \rfloor - \frac{1}{2})$. Erdös and Turán showed, for $d \in \mathbb{N}$,

$$\Delta(\{x_i\}) \leq \frac{C_0}{d} + \sum_{k=1}^{d} \frac{C_k}{M} \left| \sum_{m=1}^{M} e^{2\pi ikm} \right|.$$

- *N*-dimensional version: Let $\{\mathbf{x}_i\}_{i=1}^M$ be points in the unit sphere S^{N-1}. Then, for each $d \in \mathbb{N}$, we have

$$\sup_{|y|\leq 1} \frac{1}{M} \log \prod_{m=1}^{M} |\mathbf{y} - \mathbf{x}_m| \leq \frac{C_0}{d} + \sum_{k=1}^{d} C_k \sum_{l=1}^{\theta_k(N)} \frac{1}{M} \left| \sum_{m=1}^{M} Y_{kl}(\mathbf{x}_m) \right|.$$

Some applications (3) - Consequences of the Riemann hypothesis

Assuming RH...

- Chandee and Soundararajan '09 showed that

$$\log|\zeta(\tfrac{1}{2}+it)| \leq \frac{\log 2}{2}\frac{\log t}{\log\log t} + l.o.t.$$

as $t \to \infty$ (previous result of Soundararajan $C = 0.372$).

- C., Chandee and Milinovich '12 showed that

$$|S(t)| \leq \frac{1}{4}\frac{\log t}{\log\log t} + l.o.t.$$

as $t \to \infty$ (previous result of Goldston and Gonek $C = 1/2$).

- Similar bounds for the function $S_1(t)$ and for the pair correlation function.

Main problem (joint with F. Littmann)

- Let $\nu > -1$. Given $f : \mathbb{R}^N \to \mathbb{R}$ we want to find entire functions $L : \mathbb{C}^N \to \mathbb{C}$ and $M : \mathbb{C}^N \to \mathbb{C}$ of exponential type (with respect to the unit Euclidean ball) at most $\tau > 0$ such that

$$L(\mathbf{x}) \leq f(\mathbf{x}) \leq M(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^N$, and such that the integral

$$\int_{\mathbb{R}^N} \{M(\mathbf{x}) - L(\mathbf{x})\} \, |\mathbf{x}|^{2\nu+2-N} \, \mathrm{d}\mathbf{x}$$

is minimized.

- We shall solve this problem for a class of radial functions f of the form

$$f(\mathbf{x}) = \int_0^\infty \left\{ e^{-\lambda|\mathbf{x}|} - e^{-\lambda} \right\} \mathrm{d}\mu(\lambda)$$

where

$$\int_0^\infty \frac{\lambda}{1 + \lambda^{2\nu+3}} \mathrm{d}\lambda < \infty.$$

General strategy

- Step 1: Develop a general one-sided interpolation theory (with interpolation nodes at the zeros of Laguerre-Pólya functions) for the exponential function $f_\lambda(x) = e^{-\lambda|x|}$, for $x \in \mathbb{R}$.
- Step 2: Solve the one-dimensional extremal problem for $f_\lambda(x) = e^{-\lambda|x|}$ in the general framework of de Branges spaces.
- Step 3: Specialize the de Branges spaces to get the measure $|x|^{2\nu+1}\,\mathrm{d}x$.
- Step 4: Extend the construction to $\mathbb{R}^N$ via radial symmetrization arguments.
- Step 5: Integrate the parameter $\lambda > 0$ and use distributional arguments to extend the construction to a class of radial functions.

Step 1 - Intepolation at zeros of LP functions.

- A real entire function $F : \mathbb{C} \to \mathbb{C}$ belongs to the LP class if

$$F(z) = \frac{F^{(r)}(0)}{r!} z^r e^{-az^2+bz} \prod_{j=1}^{\infty} \left(1 - \frac{z}{x_j}\right) e^{z/x_j},$$

  where $r \in \mathbb{Z}^+$, $a, b, x_j \in \mathbb{R}$, with $a \geq 0$, $x_j \neq 0$ and $\sum_{j=1}^{\infty} x_j^{-2} < \infty$.
- The inverse of F can be expressed as a Laplace transform in each vertical strip $\tau_1 < \Re(z) < \tau_2$ not containing zeros:

$$\frac{1}{F(z)} = \int_{-\infty}^{\infty} g_c(t)\, e^{-zt}\, \mathrm{d}t.$$

- Let $\alpha_F > 0$ be the smallest positive zero of F.
- Let $\beta_F \leq 0$ the largest nonpositive zero of F.

Lemma (1)

Let F be a Laguerre-Pólya function. Let $g = g_{\alpha_F/2}$ and assume that $F(\alpha_F/2) > 0$ (in case $\alpha_F = +\infty$, let $g = g_1$ and assume $F(1) > 0$). Define, for $\lambda > 0$,

$$\mathcal{A}_1(F,\lambda,z) = F(z)\int_{-\infty}^{0} g(w-\lambda)\, e^{-zw}\,\mathrm{d}w \ \text{ for } \Re(z) < \alpha_F,$$

$$\mathcal{A}_2(F,\lambda,z) = e^{-\lambda z} - F(z)\int_{0}^{\infty} g(w-\lambda)\, e^{-zw}\,\mathrm{d}w \ \text{ for } \Re(z) > \beta_F.$$

Then $z \mapsto \mathcal{A}_1(F,\lambda,z)$ is analytic in $\Re(z) < \alpha_F$, $z \mapsto \mathcal{A}_2(F,\lambda,z)$ is analytic in $\Re(z) > \beta_F$, and these functions are restrictions of an entire function, which we will denote by $\mathcal{A}(F,\lambda,z)$. Moreover, there exists $c > 0$ so that

$$|\mathcal{A}(F,\lambda,z)| \leq c\left(1 + |F(z)|\right)$$

for all $z \in \mathbb{C}$.

Lemma (2)

Let $\lambda > 0$. *Let* F *be an even Laguerre-Pólya function such that* $F(0) > 0$. *Then the entire function* $z \mapsto L(F, \lambda, z)$ *defined by*

$$L(F, \lambda, z) = \mathcal{A}(F, \lambda, z) + \mathcal{A}(F, \lambda, -z)$$

satisfies

$$F(x)\left\{e^{-\lambda|x|} - L(F, \lambda, x)\right\} \geq 0$$

for all $x \in \mathbb{R}$ *and*

$$L(F, \lambda, \xi) = e^{-\lambda|\xi|}$$

for all $\xi \in \mathbb{R}$ *with* $F(\xi) = 0$.

Lemma (3)

Let $\lambda > 0$. Let F be an even Laguerre-Pólya function that has a double zero at the origin. Let $g = g_{\alpha_F/2}$ and assume that $F(\alpha_F/2) > 0$ (in case $\alpha_F = +\infty$, let $g = g_1$ and assume $F(1) > 0$). Then the entire function $z \mapsto M(F, \lambda, z)$ defined by

$$M(F, \lambda, z) = \mathcal{A}(F, \lambda, z) + \mathcal{A}(F, \lambda, -z) + 2g'(0)\frac{F(z)}{z^2}$$

satisfies

$$F(x)\left\{M(F, \lambda, x) - e^{-\lambda|x|}\right\} \geq 0$$

for all $x \in \mathbb{R}$ and

$$M(F, \lambda, \xi) = e^{-\lambda|\xi|}$$

for all $\xi \in \mathbb{R}$ with $F(\xi) = 0$.

Step 2 - De Branges spaces I

- For an entire function $E : \mathbb{C} \to \mathbb{C}$ we write $E^*(z) = \overline{E(\overline{z})}$.
- A Hermite-Biehler function is an entire function $E : \mathbb{C} \to \mathbb{C}$ that satisfies
$$|E^*(z)| < |E(z)|$$
for all $z \in \mathcal{U} = \{w \in \mathbb{C};\ \Re(w) > 0\}$.
- We consider the space $\mathcal{H}(E)$ of entire functions $F : \mathbb{C} \to \mathbb{C}$ such that
$$\|F\|_E^2 := \int_{-\infty}^{\infty} |F(x)|^2 \, |E(x)|^{-2} \, \mathrm{d}x < \infty \, ,$$
and such that F/E and F^*/E have bounded type and nonpositive mean type in $\mathcal{U}$. This is a Hilbert space with inner product
$$\langle F, G \rangle_E := \int_{-\infty}^{\infty} F(x) \, \overline{G(x)} \, |E(x)|^{-2} \, \mathrm{d}x.$$

Note: Mean type $v(G) = \limsup_{y \to \infty} y^{-1} \log |G(iy)|$.

Step 2 - De Branges spaces II

- For each $w \in \mathbb{C}$, the map $F \mapsto F(w)$ is a continuous linear functional on $\mathcal{H}(E)$. Therefore, there exists $z \mapsto K(w,z)$ in $\mathcal{H}(E)$ such that

$$F(w) = \langle F, K(w,\cdot)\rangle_E \,.$$

- We write

$$A(z) := \frac{1}{2}\{E(z) + E^*(z)\} \quad ; \quad B(z) := \frac{i}{2}\{E(z) - E^*(z)\},$$

then A and B are real entire functions and $E(z) = A(z) - iB(z)$.
- We have

$$\pi(z - \overline{w})K(w,z) = B(z)A(\overline{w}) - A(z)B(\overline{w}),$$

or alternatively by

$$2\pi i(\overline{w} - z)K(w,z) = E(z)E^*(\overline{w}) - E^*(z)E(\overline{w}).$$

When $z = \overline{w}$ we have

$$\pi K(\overline{z}, z) = B'(z)A(z) - A'(z)B(z).$$

Step 2 - De Branges spaces III

- We assume that the Hermite-Biehler function E satisfies:

  (P1) E has bounded type in $\mathcal{U}$;

  (P2) E has no real zeros;

  (P3) $z \mapsto E(iz)$ is a real entire function;

  (P4) $A, B \notin \mathcal{H}(E)$.

- Property (P1) implies that A and B are Laguerre-Pólya functions.

- Properties (P2) and (P3) imply that A is even with $A(0) \neq 0$ and B is odd with $B(0) = 0$ (simple zero).
- Property (P4) implies that $\{z \mapsto K(\xi, z);\ A(\xi) = 0\}$ and $\{z \mapsto K(\xi, z);\ B(\xi) = 0\}$ are complete orthogonal sets in $\mathcal{H}(E)$.

Theorem

Let $\lambda > 0$. Let E be a Hermite-Biehler function satisfying (P1)-(P4) and such that $\int_{-\infty}^{\infty} e^{-\lambda|x|}\,|E(x)|^{-2}\,\mathrm{d}x < \infty$.

(i) *If $L : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type at most $2\tau(E)$ such that $L(x) \leq e^{-\lambda|x|}$ for all $x \in \mathbb{R}$ then*

$$\int_{-\infty}^{\infty} L(x)\,|E(x)|^{-2}\,\mathrm{d}x \leq \sum_{A(\xi)=0} \frac{e^{-\lambda|\xi|}}{K(\xi,\xi)},$$

Equality happens if $L(z) = L(A^2, \lambda, z)$.

(ii) *If $M : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type at most $2\tau(E)$ such that $M(x) \geq e^{-\lambda|x|}$ for all $x \in \mathbb{R}$ then*

$$\int_{-\infty}^{\infty} M(x)\,|E(x)|^{-2}\,\mathrm{d}x \geq \sum_{B(\xi)=0} \frac{e^{-\lambda|\xi|}}{K(\xi,\xi)},$$

Equality happens if $M(z) = M(B^2, \lambda, z)$.

Step 3 - Homogeneous de Branges spaces

- Let $\nu > -1$ and define the real entire functions

$$A_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n}}{n!(\nu+1)(\nu+2)...(\nu+n)}$$

  and

$$B_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+1}}{n!(\nu+1)(\nu+2)...(\nu+n+1)}.$$

- Then

$$E(z) = A_\nu(z) - iB_\nu(z)$$

  is a Hemite-Biehler function satisfying properties (P1)-(P4) (with $v(E_\nu) = \tau(E_\nu) = 1$).
- For $F \in \mathcal{H}(E_\nu)$ we have the identity

$$\int_{-\infty}^{\infty} |F(x)|^2 \, |E_\nu(x)|^{-2} \, \mathrm{d}x = c_\nu \int_{-\infty}^{\infty} |F(x)|^2 \, |x|^{2\nu+1} \, \mathrm{d}x \,,$$

  with $c_\nu = \pi \, 2^{-2\nu-1} \, \Gamma(\nu+1)^{-2}$.

Step 4 - Radial symmetrization techniques I

Lemma (4)

Let $F : \mathbb{C} \to \mathbb{C}$ be an even entire function with power series representation

$$F(z) = \sum_{k=0}^{\infty} c_k z^{2k}$$

and let $\psi_N(F) : \mathbb{C}^N \to \mathbb{C}$ be the entire function

$$\psi_N(F)(\mathbf{z}) = \sum_{k=0}^{\infty} c_k (z_1^2 + \ldots + z_n^2)^k.$$

Then F has exponential type if and only if $\psi_N(F)$ has exponential type, and $\tau(F) = \tau(\psi_N(F))$.

Step 4 - Radial symmetrization techniques II

For $N \geq 2$ let $SO(N)$ denote the compact topological group of real orthogonal $N \times N$ matrices M with $\det M = 1$. For $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}$ we define $\widetilde{\mathcal{F}} : \mathbb{C}^N \to \mathbb{C}$ by

$$\widetilde{\mathcal{F}}(\mathbf{z}) = \int_{SO(N)} \mathcal{F}(M\mathbf{z})\, \mathrm{d}\sigma(M).$$

Lemma (5)

Let $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}$ be an entire function. Then $\widetilde{\mathcal{F}} : \mathbb{C}^N \to \mathbb{C}$ is an entire function that satisfies the following properties:

(i) *$\widetilde{\mathcal{F}}$ has a power series expansion of the form*

$$\widetilde{\mathcal{F}}(\mathbf{z}) = \sum_{k=0}^{\infty} c_k (z_1^2 + \ldots + z_n^2)^k.$$

(ii) *If $\mathcal{F}$ has exponential type then $\widetilde{\mathcal{F}}$ has exponential type and $\tau(\widetilde{\mathcal{F}}) \leq \tau(\mathcal{F})$.*

Step 5 - Integrating the parameter

- For the homogeneous spaces E_ν we can actually compute the asymptotics of the error term as a function of the parameter λ, since the zeros of A_ν and B_ν are the zeros of Bessel functions. This allows us to impose the conditions of the measure $\mathrm{d}\mu$ to integrate λ.
- The construction of majorants and minorants here uses the Paley-Wiener theorem for distributions.

Related problems

2.
   - To majorize/minorize a given function $f : [-1, 1] \to \mathbb{R}$ by polynomials of degree d, we use our interpolation method at the zeros of the orthogonal polynomials. The constructed function will be a polynomial.
   - The optimality will now come from the so called quadrature formulas (Gauss quadrature and Lobatto quadrature).

3.
   - In the case of periodic functions we use our interpolation method at the zeros of the periodic functions given by the corresponding orthogonal polynomials in the unit circle. The constructed function will be a trigonometric polynomial.
   - Optimality will come now from the theory of reproducing kernel Hilbert spaces of polynomials.
   - At the end we integrate the free parameter and extend the construction to a class of even functions.

Thank you!!