Differentiation Properties Related to the Keleti Perimeter to Area Conjecture

Paul Humke, Cameron Marcott, Bjorn Mellem, Cole Stiegler

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Keleti's Perimeter to Area Conjecture (*PAC*)

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- To date, the best known bound is slightly less than **5.6**.

Gyenes' Results

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There exist congruent convex sets , $E_1 \cong E_2 \subset \mathbb{R}^2$ such that the perimeter to area ratio for $E_1 \cup E_2$ exceeds the perimeter to area ratio for either one of them.

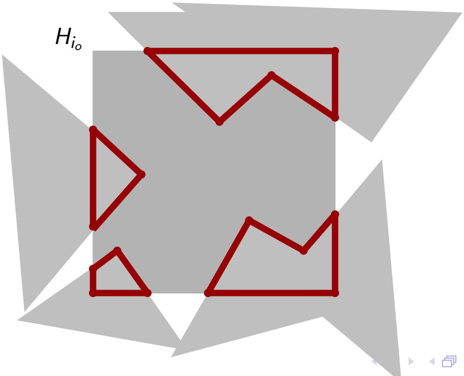
A Convex Counterexample

A Tantalizing Tidbit

Suppose a counterexample exists. Then there is a counterexample with a least number of squares. The Isoperimetric Inequality yields

Theorem

If $\mathcal{H} = \bigcup_{i=1}^{n} H_i$ is an optimal counterexample, then for each $i \leq n$, the area of $H_i \cap (\mathcal{H} \setminus H_i) > \frac{\pi}{4}$.



Basic Notation

1. $\mathcal{H} = \bigcup_{i=1}^{n} H_i$ is the finite union of unit squares H_i in $\mathbb{R}^2$.
2. $p(\mathcal{H})$ is the perimeter of $\mathcal{H}$.
3. $\alpha(\mathcal{H})$ denotes the area of $\mathcal{H}$.
4. square $\equiv$ unit square in $\mathbb{R}^2$.

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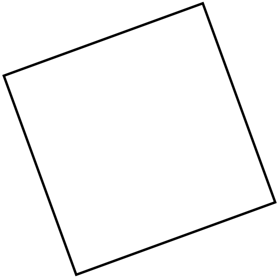

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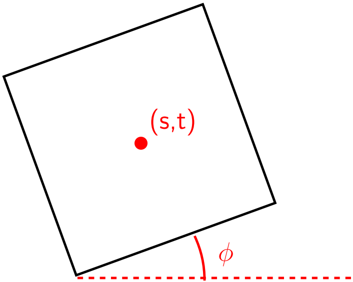

Basic Notation Revisited

Suppose we are interested in unions of n unit squares H_i; $\mathcal{H} = \bigcup_{i=1}^{n} H_i$. Then the associated perimeter and area are maps:

1. $p : \mathbb{R}^{3n} \longrightarrow \mathbb{R}$.
2. $\alpha : \mathbb{R}^{3n} \longrightarrow \mathbb{R}$.
3. $\kappa \equiv \frac{p}{\alpha}$.

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We'll have a brief look at some continuity and differentiability of these maps.

Continuity of p and α

Now, α IS the area function afterall, Lipschitz in each coordinate, so . . .

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α *is continuous.*

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Discontinuity of p

DISCONTINUITY OF p

Consequently, let's first restrict the domain somewhat to avoid such unpleasantries.

1. $\mathcal{H} \subset \mathbb{R}^2$ has **distinct rotational displacement** if $\phi_i \neq \phi_j$ when $i \neq j$

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Theorem (a brief aside)

The set of points which are in standard position is the complement of a sparse set *in the sense that it is a subset of the complement of a countable union of monotonic surfaces and so are both residual and of full measure in* $\mathbb{R}^{3n}$.

Continuity of p and α

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The perimeter function p is continuous at every point $\mathcal{H} \in \mathbb{R}^{3n}$ which is in standard position.

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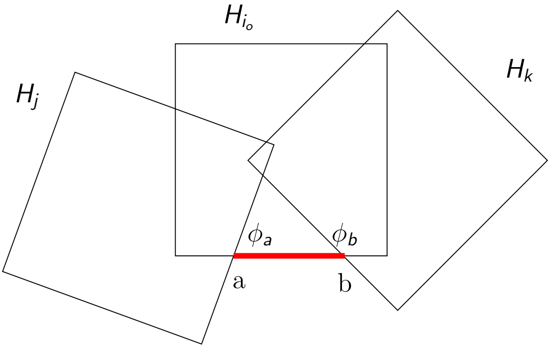

Differentiability of p and α

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So we do the obvious:

1. **compute the first partials** and
2. **show they're continuous.**

Differentiability of p

Theorem

The perimeter function $p : \mathbb{R}^{3n} \to \mathbb{R}^{+}$ is differentiable at every point $\mathcal{H} \in \mathbb{R}^{3n}$ in standard position.

Outline of Proof.

1. Fix $\mathcal{H} \in \mathbb{R}^{3n}$ and $1 \leq i_o \leq n$.

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$\frac{\partial p}{\partial \phi_{i_o}}$ for example.

$$\frac{\partial p}{\partial \phi_{i_o}}$$

CASE 1. The segment misses H_{i_o}.

In this case the contribution is **0**.

CASE 2. The segment lies on H_{i_o}.

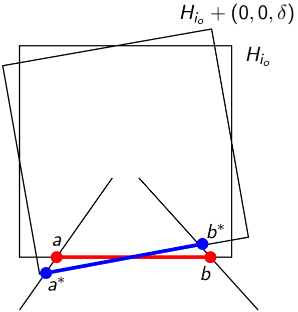


Figure: $[a, b]$ and $[a^*, b^*]$

Case 2. Computation

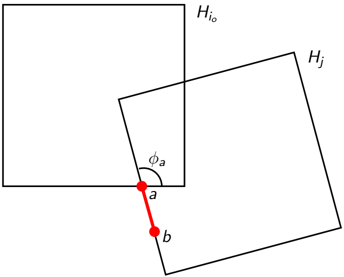
$$a^* = \left(\frac{x_1 \tan\phi_a}{\tan\phi_a - \tan\delta}, \frac{x_1 \tan\phi_a \tan\delta}{\tan\phi_a - \tan\delta} - \frac{1}{2}\right)$$
$$b^* = \left(\frac{x_2 \tan\phi_b}{\tan\phi_b + \tan\delta}, \frac{x_2 \tan\phi_b \tan\delta}{\tan\phi_b + \tan\delta} - \frac{1}{2}\right).$$

Hence, **with some trigonometry and limit taking:**

$$\lim_{\delta\to 0} \frac{\left|b^* - a^*\right| - \left|b - a\right|}{\delta} = \left|\mathbf{b} - \mathbf{a}\right|(\cot\phi_{\mathbf{b}} - \cot\phi_{\mathbf{a}}).$$

Case 3.

Case 3. The segment intersects H_{i_o} but does not lie on it.



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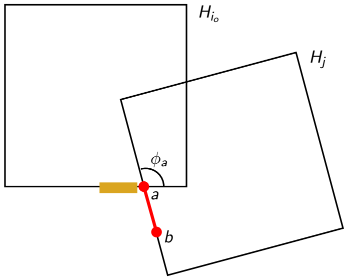

Again, **with some trigonometry and limit taking:**

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Oh yes, here is "d."

$$\frac{\partial p}{\partial s_{i_o}} \text{ and } \frac{\partial p}{\partial t_{i_o}}$$

These cases have **congruent geometries** and are handled similarly to the case of $\frac{\partial p}{\partial \phi_{i_o}}$.

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These cases have **congruent geometries** and are handled similarly to the case of $\frac{\partial p}{\partial \phi_{i_o}}$.

But we still have area to deal with.

Differentiability of α

Theorem

The area function $\alpha : \mathbb{R}^{3n} \to \mathbb{R}^{+}$ is differentiable at every point $\mathcal{H} \in \mathbb{R}^{3n}$ in standard position.

Outline of Proof.

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Differentiability of α

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The area function $\alpha : \mathbb{R}^{3n} \to \mathbb{R}^{+}$ *is differentiable at every point* $\mathcal{H} \in \mathbb{R}^{3n}$ *in standard position.*

Outline of Proof.

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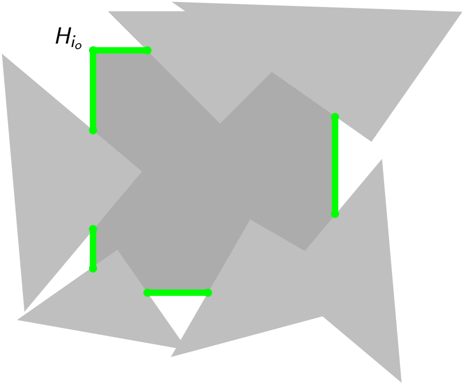
$\frac{\partial \alpha}{\partial \phi_{i_o}}$ for example.

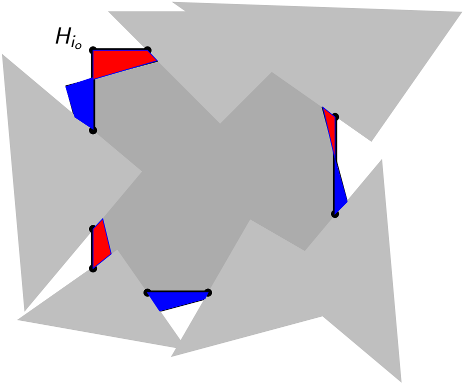
$\mathcal{H}$ with H_{i_o} Darkened

$$\frac{\partial \alpha}{\partial \phi_{i_o}}$$

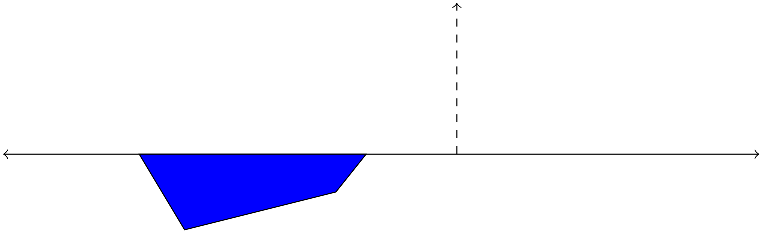


H with H_{i_o} and Rotated H_{i_o}

$$\frac{\partial \alpha}{\partial \phi_{i_o}}$$



Pre Computations; What are the Variables?



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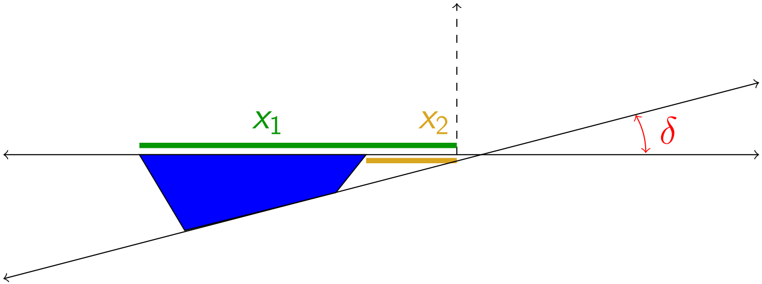

The Computations

$$\Delta\alpha([a,b]) = \frac{({x_1}^2 - {x_2}^2)\tan\phi_a \tan\phi_b \tan\delta + \tan^2\delta({x_2}^2\tan\phi_b + {x_1}^2\tan\phi_a)}{2(\tan\phi_a - \tan\delta)(\tan\phi_b + \tan\delta)}.$$

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$$\frac{\partial\alpha}{\partial\phi_{i_o}} \text{ at [a,b] } = \lim_{\delta\to 0}\frac{\Delta\alpha([a,b])}{\delta} = \frac{{x_1}^2 - {x_2}^2}{2}.$$

$$\frac{\partial \alpha}{\partial s_{i_o}} \text{ and } \frac{\partial \alpha}{\partial t_{i_o}}$$

These cases again have **congruent geometries** and are handled similarly.

Where We're Pushing the Pebble

Similar ground has been plowed in other lands. For example:

Theorem (Kneser-Paulson)

If a finite set of discs are rearranged so that the distance between the centers of any pair decreases, then the area and the perimeter of the union of the discs also decreases.

THANK YOU!

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