Almost automorphic solutions of dynamic equations on time scales

Jaqueline Godoy Mesquita

Joint work with Carlos Lizama

Universidade de São Paulo - USP

Instituto de Ciências Matemáticas e de Computação

Supported by CAPES

June, 2013

Historical overview

1. 1988 - Stefan Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, PhD thesis, Universität Würzburg.

Historical overview

1. 1988 - Stefan Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, PhD thesis, Universität Würzburg.

Historical overview

1. 1988 - Stefan Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, PhD thesis, Universität Würzburg.

Historical overview

1. 1988 - Stefan Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, PhD thesis, Universität Würzburg.

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$);
4. Applications to economic problems;
5. Applications to population models;

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$);
4. Applications to economic problems;
5. Applications to population models;

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$);
4. Applications to economic problems;
5. Applications to population models;

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$);
4. Applications to economic problems;
5. Applications to population models;

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$); V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, 2001.
4. Applications to economic problems;
5. Applications to population models;

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$);
4. Applications to economic problems;
5. Applications to population models;

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$);
4. Applications to economic problems;
   F. M. Atici, D. C. Biles, A. Lebedinsky, An application of time scales to economics, *Mathematical and Computer Modelling*, 43, 718-726 (2006)
5. Applications to population models;

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$);
4. Applications to economic problems;
5. Applications to population models;

Motivation to study dynamic equations on time scales:

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time-scale set;
3. Applications to quantum calculus (when $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$);
4. Applications to economic problems;
5. Applications to population models;
   F. B. Christiansen and T. M. Fenchel, *Theories of populations in biological communities*, vol. 20 of Lectures Notes in Ecological Studies, Springer-Verlag, Berlin, 1977

Time scales calculus

Definition

A time scale is a nonempty and closed subset of the real numbers.

Example

$\mathbb{Z}$, $\mathbb{R}$, Cantor set, closed intervals, among others are examples of time scales.

Notation: $\mathbb{T}$, $[a, b]_{\mathbb{T}}$.

Time scale - Basic Concepts

Definition

Define the following operators:

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ and } \sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

where ρ is called *backward jump operator* and σ is called *forward jump operator*.

Definitions	t
right-dense	$\sigma(t) = t$
right-scattered	$\sigma(t) > t$
left-dense	$\rho(t) = t$
left-scattered	$\rho(t) < t$

Example

If $\mathbb{T} = \mathbb{R}$, then, for $t \in \mathbb{R}$, we obtain

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, +\infty) = t,$$

$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t,$$

Thus, t is right-dense and left-dense at the same time.

Example

If $\mathbb{T} = \mathbb{Z}$, then for any $t \in \mathbb{Z}$, we have

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, t+3, \ldots\} = t+1 > t,$$

$$\rho(t) = \sup\{s \in \mathbb{Z} : s < t\} = \sup\{t-1, t-2, t-3, \ldots\} = t-1 < t.$$

Thus, t is right-scattered and left-scattered at the same time.

Definition

We define the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by

$$\mu(t) := \sigma(t) - t.$$

Examples

1. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, for all $t \in \mathbb{T}$.
2. If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$, for all $t \in \mathbb{T}$.

Definition

Define the set $\mathbb{T}^\kappa$ as follows

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases} \tag{1}$$

Definition

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, define $f^{\Delta}(t)$ to be the number (provided it exists) satisfying $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$. We call $f^{\Delta}(t)$ the *delta* (or *Hilger*) *derivative* of f at t.

Properties of delta-derivatives

(i) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(ii) If t is right-dense, then f is delta-differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

Examples

(i) If $\mathbb{T} = h\mathbb{Z}$, then for a function $f : \mathbb{T} \to \mathbb{R}$, we get

$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h},$$

for all $t \in \mathbb{T}$.

(ii) If $\mathbb{T} = \mathbb{R}$, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

for every $t \in \mathbb{T}$.

Definition

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$.

Definition

A $m \times n$ matrix-valued function A on a time scale $\mathbb{T}$ is called *regressive* on $\mathbb{T}$ provided

$$I + \mu(t)A(t) \text{ is invertible for all } t \in \mathbb{T}^k,$$

and the class of all such regressive rd-continuous is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{m\times n})$.

Definition

Let $A \in C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$. We say that the linear system

$$x^{\Delta}(t) = A(t)x(t) \tag{2}$$

has an *exponential dichotomy* on $\mathbb{T}$ if there exist positive constants K and γ, projection P such that the fundamental matrix $X(t)$ of (2) satisfies

$$|X(t)PX^{-1}(s)| \leq Ke_{\ominus\gamma}(t,s), \quad s,t \in \mathbb{T}, \ \ t \geq s,$$

$$|X(t)(I-P)X^{-1}(s)| \leq Ke_{\ominus\gamma}(s,t), \quad s,t \in \mathbb{T}, \ \ t \leq s.$$

A new class of time scales

Definition

A time scale $\mathbb{T}$ is called *invariant under translations* if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \ \forall t \in \mathbb{T}\} \neq \{0\}. \quad (3)$$

Properties:

1. If $\tau_1, \tau_2 \in \Pi$, then $\tau_1 \pm \tau_2 \in \Pi$.
2. If $\mathbb{T}$ is an invariant under translations time scale, then

$$\inf \mathbb{T} = -\infty \quad \text{and} \quad \sup \mathbb{T} = +\infty.$$

Examples of invariant under translations time scales

1. $\mathbb{R}$;
2. h $\mathbb{Z}$, $h \in \mathbb{Z}$;
3. $\frac{1}{n}\mathbb{Z}$, $n \in \mathbb{N} \setminus \{0\}$;
4. $\mathbb{P}_{a,b} = \bigcup_{k=-\infty}^{\infty} [k(a+b), k(a+b)+a]$, $a \neq b$;

Properties of invariant under translations time scales:

(i) If t is right-dense, then for every $h \in \Pi$, $t+h$ is right-dense.

(ii) If t is right-scattered, then for every $h \in \Pi$, $t+h$ is right-scattered.

(iii) If $h \in \Pi$, then

$$\sigma(t)+h=\sigma(t+h) \quad \text{and} \quad \sigma(t)-h=\sigma(t-h),$$

for every $t \in \mathbb{T}$.

(iv) If $h \in \Pi$, then

$$\mu(t+h)=\mu(t)=\mu(t-h)$$

for every $t \in \mathbb{T}$.

Definition

Let X be a Banach space and $\mathbb{T}$ be an invariant under translation time scale. Then, an rd-continuous function $f : \mathbb{T} \to X$ is called *almost automorphic* on $\mathbb{T}$ if for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} f(t+\alpha_n) = \bar{f}(t)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{f}(t-\alpha_n) = f(t),$$

for every $t \in \mathbb{T}$.

Notation: $AA_{\mathbb{T}}(X)$.

Properties

Let $\mathbb{T}$ be invariant under translations and $f, g \in AA_{\mathbb{T}}(X)$, then

(i) $f + g \in AA_{\mathbb{T}}(X)$;

(ii) $cf \in AA_{\mathbb{T}}(X)$ for every scalar c;

(iii) $gf \in AA_{\mathbb{T}}(X)$;

(iv) $\sup_{t\in\mathbb{T}} \|f(t)\| < \infty$.

Theorem

If $f_n \in AA_{\mathbb{T}}(X)$, $\forall n \in \mathbb{N}$ and $f_n \to f$ uniformly, then $f \in AA_{\mathbb{T}}(X)$.

Applications to dynamic equations on time scales

Almost automorphic solutions of dynamic equations on time scales

- Existence of almost automorphic solutions of linear dynamic equations on time scales given by

$$x^{\Delta}(t) = A(t)x(t) + f(t)$$

- Existence and uniqueness of almost automorphic solutions of linear dynamic equations on time scales given by

$$x^{\Delta}(t) = A(t)x(t) + f(t, x)$$

D. Araya, R. Castro, C. Lizama
Almost automorphic solutions of difference equations, Advances in Difference Equations, Vol 2009, Article ID 591380, 15 pages.

S. Bochner
Continuous mappings of almost automorphic and almost periodic functions, Proc. Nat. Acad. Sci. USA, 52 (1964), 907-910.

S. Bochner
Uniform convergence of monotone sequences of functions, Proc. Nat. Acad. Sci. USA, 47 (1961), 582-585.

S. Bochner
A new approach in almost-periodicity, Proc. Nat. Acad. Sci. USA, 48 (1962), 2039-2043.

References

M. Bohner and A. Peterson
Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

G. M. N'Guérékata
Almost Automorphic and Almost Periodic Functions in Abstract Spaces, Kluwer Academic/Plenium Publishers, New York, 2001.

C. Lizama, J. G. Mesquita
Almost automorphic solutions of dynamic equations on time scales, submitted.

Thanks for your attention!