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Extensions of differentiable functions

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Sugarcane Symposium, 3.6.2013 - 6.6.2013

Fréchet derivative & Strict derivative in $\mathbb{R}^n$

- We identify $\mathbb{R}^n$ with its dual space $(\mathbb{R}^n)^*$ of all linear functionals on $\mathbb{R}^n$.

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Definition (Derivative & Strict derivative)

Let $\emptyset \neq F \subset \mathbb{R}^n$ be a closed set, $f : F \to \mathbb{R}$ a function and $a \in F$.
We say that $L_a \in \mathbb{R}^n$ (resp. $S_a \in \mathbb{R}^n$) is a *(Fréchet) derivative* (resp. *strict derivative*) *of f at a* (with respect to F) if either
a is an isolated point of F, or

$$\lim_{\substack{y \to a \\ y \in F}} \frac{f(y) - f(a) - L_a \cdot (y - a)}{|y - a|} = 0$$

$$\left(\text{resp.} \quad \lim_{\substack{y \to a \\ x \to a \\ x, y \in F,\ x \neq y}} \frac{f(y) - f(x) - S_a \cdot (y - x)}{|y - x|} = 0\right).$$

Extension theorem with preservation of points of continuity of the derivative

Theorem (M. Koc, L. Zajíček (2012))

Let $F \subset \mathbb{R}^n$ be a nonempty closed set, $f : F \to \mathbb{R}$ a function and $L : F \to \mathbb{R}^n$ a derivative of f such that $L \in \mathcal{B}_1(F)$.
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- Joint generalization of the extension theorem with preservation of the derivative [V. Aversa, M. Laczkovich, D. Preiss (1985)] and Whitney's $\mathcal{C}^1$ extension theorem [H. Whitney (1934)].

Result on extensions of $\mathcal{B}_1$ functions

Theorem (M. Koc, L. Zajíček (2012))

Let (X, ϱ) be a metric space, $H \subset X$ a nonempty closed set and $b : H \to \mathbb{R}$ a Baire one function on H. Then there exists a function $g : (X \setminus H) \to \mathbb{R}$ such that

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- To prove an extension theorem for vector-valued functions, we need an analogue of the previous theorem for vector-valued $\mathcal{B}_1$ functions.

Result on extensions of $\mathcal{B}_1$ functions (vector-valued version)

Theorem (M. Koc, J. Kolář (2013))

Let (X, ϱ) be a metric space, $H \subset X$ a nonempty closed set, Z a normed linear space and $b : H \to Z$ a Baire one function on H. Then there exists a continuous function $g : (X \setminus H) \to Z$ such that

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(iii) *g is bounded on a neighborhood of a (with respect to $X \setminus H$) whenever $a \in \partial H$ and b is bounded on a neighborhood of a (with respect to H).*

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Definition (Fréchet derivative & Strict derivative)

Let X and Y be normed linear spaces, $A \subset X$ an arbitrary set, $f : A \to Y$ a function and $a \in A$.

A bounded linear operator $F_a : X \to Y$ is called a *relative Fréchet derivative of f at a* (with respect to A) if either a is an isolated point of A, or

$$\lim_{\substack{x \to a \\ x \in A}} \frac{\|f(x) - f(a) - F_a(x - a)\|_Y}{\|x - a\|_X} = 0.$$

A bounded linear operator $S_a : X \to Y$ is called a *relative strict derivative of f at a* (with respect to A) if either a is an isolated point of A, or

$$\lim_{\substack{y \to a \\ x \to a \\ x, y \in A,\ x \neq y}} \frac{\|f(y) - f(x) - S_a(y - x)\|_Y}{\|y - x\|_X} = 0 \quad (x = a,\ y = a \text{ allowed}).$$

Extensions preserving points of continuity, differentiability ... (vector-valued version)

Theorem (M. Koc, J. Kolář (2013))

Let $F \subset \mathbb{R}^n$ be a nonempty closed set, Y a normed linear space, $f : F \to Y$ an arbitrary function and $L : F \to \mathcal{L}(\mathbb{R}^n, Y)$ a function that is Baire one on F. Then there exists a function $\bar{f} : \mathbb{R}^n \to Y$ such that

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Corollary

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- It easily implies the $\mathcal{C}^1$-case of Whitney's Extension Theorem for vector-valued functions.

Extending from infinite-dimensional domains

Theorem (M. Koc, J. Kolář (2013))

Let $p \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. Let X be a normed linear space that admits a $\mathcal{C}^p$-smooth partition of unity, $F \subset X$ a nonempty closed set, Y a normed linear space, $f : F \to Y$ an arbitrary function and $L : F \to \mathcal{L}(X, Y)$ a function that is Baire one on F. Then there exists a function $\bar{f} : X \to Y$ such that

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(iv) *if* $a \in F$ *and* $L(a)$ *is a relative Fréchet derivative of* f *at* a *(with respect to* F*), then* $(\bar{f})'(a) = L(a)$,

(v) $\bar{f} \in \mathcal{C}^p(X \setminus F, Y)$.

Corollary

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Corollary (M. Koc, J. Kolář (2013))

Let X be a normed linear space that admits a $\mathcal{C}^1$-smooth partition of unity, $F \subset X$ a nonempty closed set, Y a normed linear space, $f : F \to Y$ an arbitrary function, $L : F \to \mathcal{L}(X, Y)$ a relative Fréchet derivative of f (on F) such that L is a Baire one function on F. Then there exists a function $\bar{f} : X \to Y$ such that $\bar{f}$ extends f, $\bar{f}$ is Fréchet differentiable everywhere on X and $(\bar{f})' = L$ on F.

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- Let $p \in \mathbb{N} \cup \{\infty\}$. It seems to be an open problem whether every Banach space that admits a $\mathcal{C}^p$-smooth bump must also admit a $\mathcal{C}^p$-smooth partition of unity. It holds for separable spaces and more generally for subspaces of WCG Banach spaces, for Banach spaces whose dual is WCG, for WCD Banach spaces, for duals of Asplund spaces and even for Banach spaces with a separable projective resolution of identity.

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- Spaces $c_0(\Gamma)$ with arbitrary set Γ and all Hilbert spaces admit $\mathcal{C}^\infty$-smooth partitions of unity, L_p spaces admit higher order smooth partitions of unity and any reflexive space admits $\mathcal{C}^1$-smooth partitions of unity [H. Toruńczyk (1973)].

References

V. Aversa, M. Laczkovich, D. Preiss
Extension of differentiable functions
Comment Math. Univ. Carolin. 26 (1985), 597–609.

M. Koc, J. Kolář
Extensions of differentiable functions (in preparation).

M. Koc, L. Zajíček
A joint generalization of Whitney's $\mathcal{C}^1$ extension theorem and Aversa-Laczkovich-Preiss' extension theorem
J. Math. Anal. Appl. 388 (2012), 1027–1039.

H. Toruńczyk
Smooth partitions of unity on some nonseparable Banach spaces
Studia Math. 46 (1973), 43–51.

H. Whitney
Differentiable functions defined in closed sets I
Trans. Amer. Math. Soc. 36 (1934), 369–387.