The Kurzweil-Henstock integral and its extensions : a historical survey

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This lecture is dedicated to the memory of my friend

STEFAN SCHWABIK,

an enthusiastic ambassador of the Kurzweil-Henstock integral,

and a great friend of many mathematicians of Sao Carlos

I. A short history of integration

Cauchy

- 1823 : *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*
- *"In the integral calculus, it seemed to me necessary to prove in a general way the existence of the integral of primitivable functions before letting their various properties to be known.*
  To reach this aim, it was first necessary to establish the notion of integral taken between given limit or definite integrals.
  As those last ones can be sometimes infinite or undeterminated, it was essential to search in which case they keep a unique and finite value"

Integral of a continuous function

- $f : [a, b] \to \mathbb{R}$ continuous
- **P-partition** of $[a, b] : \Pi := (x^j, I^j)_{1 \le j \le m}, \; I^j = [a_{j-1}, a_j]$
  $a = a_0 < a_1 < \ldots < a_{m-1} < a_m = b, \; x^j \in I^j$
- **length** of $I^j : |I^j| = a_j - a_{j-1}$
  mesh of $\Pi \; : \; M(\Pi) = \max_{1 \le j \le m} |I^j|$

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- 22-23th lectures : f *continuous on* $[a,b] \Rightarrow \exists\,!\, J \in \mathbb{R}$,
  $\forall\, \varepsilon > 0, \exists\, \eta > 0, \forall\, \Pi : M(\Pi) \le \eta : |J - \sum_{j=1}^{m} f(x^j)|I^j|| \le \varepsilon$
- $J = \int_a^b f(x)\, dx$: **definite integral** of f on $[a,b]$
- *continuity on* $[a,b] \Leftrightarrow$ *uniform continuity on* $[a,b]$

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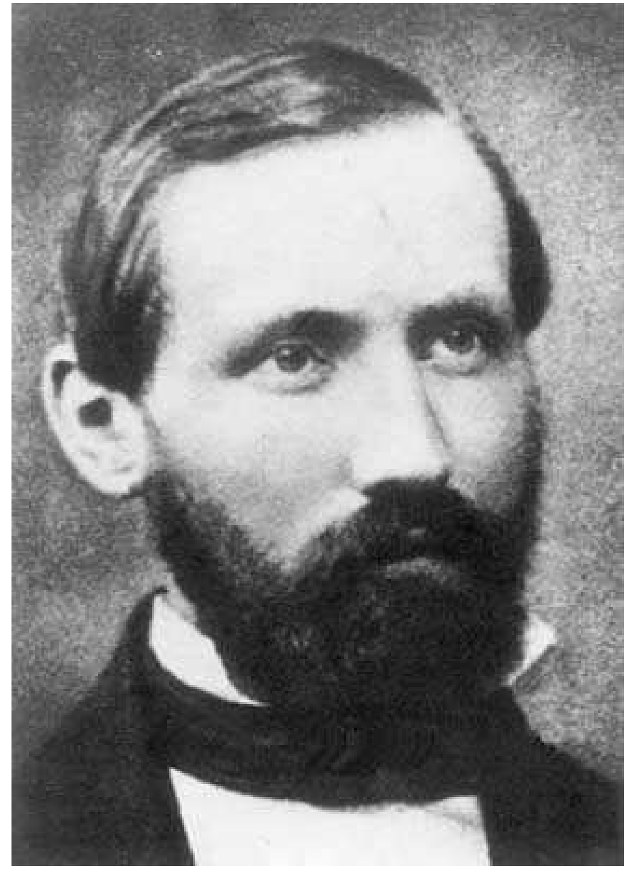
- $f \in C^1([a,b]) \Rightarrow \int_a^b f' = f(b) - f(a)$
- $f \in C([a,b]) \Rightarrow \int_a^{\cdot} f \in C^1([a,b]),\ \left(\int_a^x f\right)' = f(x)$

Cauchy and Riemann

AUGUSTIN CAUCHY

1789–1857



BERNHARD RIEMANN

1826–1866

Riemann

- 1854 : *Habilitation* thesis University of Göttingen (published 1867)
- *"The uncertainty which still prevails on some fundamental points of the theory of definite integrals forces us to place here a few remarks on the notion of definite integral, and on its possible generality. And first, what do we mean by $\int_a^b f(x)\,dx$?"*

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- $f : [a,b] \to \mathbb{R}$ is **R-integrable** on $[a,b]$ if $\exists\, J \in \mathbb{R}$, $\forall\, \varepsilon > 0, \exists\, \eta > 0, \forall\, \Pi, M(\Pi) \leq \eta : |J - \sum_{j=1}^{m} f(x^j)|I^j|| \leq \varepsilon$
- $S(f,\Pi) := \sum_{j=1}^{m} f(x^j)|I^j|$: **Riemann sum** for f and Π
- $J = \int_a^b f$ is the **R-integral** of f on $[a,b]$
- R-integrable functions are the ones for which CAUCHY's limit process made for continuous functions works

Range of R-integration

- RIEMANN : *"Let us search now the range and the limit of the preceding definition and let us ask the question :
in which case is a function integrable ?
And in which case not integrable ?"*

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- although modeled on CAUCHY's process for (uniformly) continuous functions, R-integrable functions may have a *dense* set of discontinuities
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- however, $1_{\mathbb{Q}}$ *is not R-integrable on any interval*
- R-integrable functions are characterized in terms of some 'measure' of their set of discontinuities
- indefinite R-integral of f not differentiable at points of discontinuity of f
- $\exists$ bounded derivatives not R-integrable (VOLTERRA)

Lebesgue

- 1902 : PhD thesis, *Annali di Mat. Pura Appl.*
- *“In the case of continuous functions, the notions of [indefinite] integral and of primitive are identical.*
  Riemann has defined the integral of some discontinuous functions, but all derivatives are not integrable in Riemann sense.
  The problem of the primitive functions is therefore not solved by [R-]integration, and one can wish to have a definition of the integral containing as special case that of Riemann and solving the problem of the primitives”

L-integral

- based upon a concept of **measure** of a bounded set $A \subset \mathbb{R}$ introduced by BOREL and developed by LEBESGUE
- **outer measure** $\mu_e(A)$ *of* $A \subset [c,d]$ *:* $\inf \sum_{j=1}^{\infty}(d_j - c_j)$ for all sequences $\{[c_j, d_j]\}_{j\in\mathbb{N}} : A \subset \cup_{j=1}^{\infty}[c_j, d_j]$
- **inner measure** $\mu_i(A) = (d-c) - \mu_e([c,d] \setminus A)$
- A **measurable** *:* $\mu_e(A) = \mu_i(A)$ *(***measure** $\mu(A)$ of A *)*

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- A **measurable** *:* $\mu_e(A) = \mu_i(A)$ *(***measure** $\mu(A)$ of A *)*
- $f : [a,b] \to \mathbb{R}$ bounded is **L- integrable** on $[a,b]$ if $\forall\, c < d$ *in range of* f, $f^{-1}([c,d))$ is measurable
- $\exists\, J \in \mathbb{R}, \forall\, \varepsilon > 0, \exists\, \eta > 0, \forall$ *P-partition* $\Pi = (y_j, [b_{j-1}, b_j])_{1 \leq j \leq m}$ *of* $[\inf_{[a,b]} f, \sup_{[a,b]} f]$, $M(\Pi) \leq \eta :$ $|J - \sum_{j=1}^{m} y_j \mu\left[f^{-1}([b_{j-1}, b_j))\right]| \leq \varepsilon$
- $J = \int_a^b f(x)\,dx$ is the **L-integral** of f on $[a,b]$

Borel and Lebesgue

ÉMILE BOREL

1871–1956

HENRI LEBESGUE

1875–1941

Comparing the R- and L- integrals

- approximating sums depend upon *measure theory*
- f *R-integrable* $\Leftrightarrow$ μ*(set of discontinuities of* f*)* $= 0$
- f *differentiable on* $[a, b]$ *,* f' *bounded* $\Rightarrow$ $\int_a^b f' = f(b) - f(a)$
- f *L-integrable on* $[a, b]$ $\Rightarrow$ $\int_a^{\cdot} f$ *differentiable with derivative* f *outside of a subset of* $[a, b]$ *of measure zero*

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- extension to *unbounded* functions
- f *R- or L-integrable* $\Rightarrow$ $|f|$ *R- or L-integrable*
- f *primitivable on* $[a,b]$ *is L-integrable on* $[a,b] \Leftrightarrow F$ *has bounded variation on* $[a,b]$
- $f(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$ *if* $x \neq 0, f(0) = 0$
  $f = F'$ *with* $F(x) x^2 \sin \frac{1}{x^2}$ *if* $x \neq 0$, $F(0) = 0$
  f *is not L-integrable near* 0

Denjoy-Perron integral

- 1912 : DENJOY (transfinite induction argument from L-integral) :
  D-integral integrating all derivatives
- 1914 : PERRON (inspired by DE LA VALLÉE-POUSSIN's characterization of L-integrability) :
  P-integral integrating all derivatives
- $F_+ [F_-]$ **over-function [under-function]** of f on $[a, b]$ if
  $F_\pm(a) = 0, \; F'_+(x) \geq f(x) \; [F'_-(x) \leq f(x)] \; \forall \, x \in [a, b]$
- f **P-integrable** on $[a, b] : \sup_{F_-} F_-(b) = \inf_{F_+} F_+(b)$
- common value = **P-integral** of f on $[a, b]$

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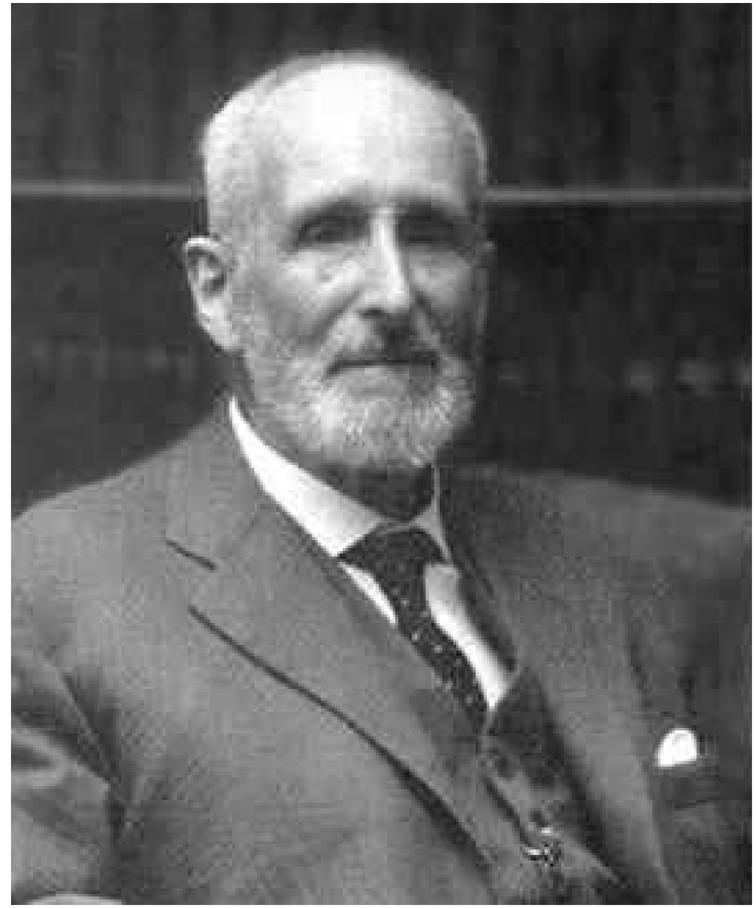
- f **P-integrable** on $[a, b] : \sup_{F_-} F_-(b) = \inf_{F_+} F_+(b)$
- common value = **P-integral** of f on $[a, b]$
- f *D-integrable on* $[a, b]$ $\Leftrightarrow$ f *P-integrable on* $[a, b]$
- f *L-integrable on* $[a, b]$ $\Leftrightarrow$ f *and* $|f|$ *DP-integrable on* $[a, b]$
- first half of XXth century : many equivalent definitions of L- and DP-integral

Denjoy and Perron

ARNAUD DENJOY

1884–1974



OSKAR PERRON

1880–1975

KH-integral

- 1957 : KURZWEIL, new definition of P-integral of $f : [a,b] \to \mathbb{R}$
- f **K-integrable** on $[a,b] : \exists\, J \in \mathbb{R},\ \forall\, \varepsilon > 0, \exists\, \delta : [a,b] \to \mathbb{R}^+,$
$\forall\, \Pi,\ x_j - \delta(x_j) \le a_{j-1} < a_j \le x_j + \delta(x_j)\ \ (1 \le j \le m),$
$|J - S(f,\Pi)| \le \varepsilon$
- Π called δ**-fine**, δ called **gauge** on $[a,b]$
- *K-integral* $\Leftrightarrow$ *P-integral*

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- Π called δ**-fine**, δ called **gauge** on $[a,b]$
- *K-integral* $\Leftrightarrow$ *P-integral*
- 1961 *:* independent rediscovery by HENSTOCK
- HENSTOCK gives many generalizations and applications
- $J = \int_a^b f$ **Kurzweil-Henstock** or **KH-integral** or **gauge integral** of f on $[a,b]$
- constant *gauge in KH-definition* $\Leftrightarrow$ *R-integral*

Henstock and Kurzweil

RALPH HENSTOCK JAROSLAV KURZWEIL

1923-2007 born in *1928*

δ-fine P-partitions

- *constant* gauge δ : δ-fine P-partition easily *constructed*
- *arbitrary* gauge δ : *existence* of a δ-fine P-partition has to be proved
- 1895 *:* done by COUSIN in a different context *(***Cousin's lemma***)*
- equivalent to the **Borel-Lebesgue property** (1894, 1902) for a compact interval
- proof depends upon the non-empty intersection property of a nested sequence of closed intervals

II. A ‘history-fiction’ of integration

Another road for Cauchy

- CAUCHY*'s aim :* construct integral calculus for *derivatives* (fundamental objects in NEWTON-LEIBNIZ's calculus)
- mimick CAUCHY's approach for *continuous* functions
- $f : [a,b] \to \mathbb{R}$ *differentiable,* with *derivative* $f' : [a,b] \to \mathbb{R}$
- $\forall\, \varepsilon > 0, \forall\, x \in [a,b], \exists\, \delta(x) > 0, \forall y \in [a,b], |y-x| \leq \delta(x) :$ $|f(y) - f(x) - f'(x)(y-x)| \leq \varepsilon |y-x|/(b-a)$
- $|f(z) - f(y) - f'(x)(z-y)| \leq \varepsilon(z-y)/(b-a)$ *if* $x - \delta(x) \leq y \leq x \leq z \leq x + \delta(x)$
- Π $\delta -$ *fine* $\Rightarrow$ $|f(a_j) - f(a_{j-1}) - f'(x_j)(a_j - a_{j-1})|$ $\leq \varepsilon(a_j - a_{j-1})/(b-a)$ $(1 \leq j \leq m)$
- Π $\delta -$ *fine* $\Rightarrow$ $|f(b) - f(a) - S(f', \Pi)| \leq \varepsilon$
- δ non constant because differentiability on $[a,b]$ $\not\Rightarrow$ *uniform* differentiability *on* $[a,b]$

Cauchy, Riemann, Weierstrass ?

- "CAUCHY" : $f : [a, b] \to \mathbb{R}$ *differentiable* $\Rightarrow \forall\, \varepsilon > 0, \exists$ *gauge* δ *on* $[a, b], \forall\, \delta$*-fine* $\Pi : |f(b) - f(a) - S(f', \Pi)| \leq \varepsilon$
- "RIEMANN" : $f : [a, b] \to \mathbb{R}$ is **integrable** on $[a, b]$ if $\exists\, J \in \mathbb{R}$, $\forall\, \varepsilon > 0, \exists$ *gauge* δ *on* $[a, b], \forall\, \delta$*-fine* $\Pi : |J - S(f, \Pi)| \leq \varepsilon$
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- $J = \int_a^b f$ **KH-integral** of f on $[a, b]$
- existence of δ-fine P-partition : CAUCHY ? RIEMANN ? WEIERSTRASS ?
- if yes, BOREL-LEBESGUE-COUSIN lemma, Denjoy-Perron's integral and Borel-Lebesgue's measure of a bounded set of $\mathbb{R}$ could have arrived half a century before

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- tragical consequence : DENJOY, PERRON, KURZWEIL and HENSTOCK disappear in our fiction : DPKH-integral is just the integral defined by "RIEMANN"

Qualities and defects of KH-integral

- qualities :
  - $\int_a^b f' = f(b) - f(a)$ *for all differentiable* f
  - *improper integrals are real integrals (*HAKE**'s theorem***)*
  - *monotone* and *dominated convergence theorems* (nice proof by HENSTOCK)
  - $E \subset [a,b]$ **measurable** *:* 1_E integrable on $[a,b]$
    measure $\mu(E) := \int_a^b 1_E$
  - *change of variable theorem*

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  - *change of variable theorem*
- defects :
  - **restriction property** holds only for *finite families of non-overlapping subintervals,* may already fail for a *countable* union of such intervals
  - due to the **non-absolute** character of the integral

We must save the soldier Lebesgue

- wanted : an integral with better **restriction property**
- f **L-integrable** on $[a,b]$ if f and $|f|$ are integrable on $[a,b]$
- f *L-integrable on* $[a,b] \Rightarrow f$ *L-integrable on any measurable* $E \subset [a,b]$
- integrability of an *unbounded derivative may be lost*
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- Hake's property may be lost (there exists improper L-integrals)
- can attribute to LEBESGUE the introduction and emphasis on this important subclass of integrable functions
- *absolute* character makes it a better tool for functional analysis (**Lebesgue spaces** $L^p(a, b)$ are Banach spaces)

III. Higher dimensions

n-dimensional KH-integral

- (closed) **n-interval** $I = I_1 \times \ldots \times I_n$, $|I|$ **n-volume** of I
- **P-partition** of I : $\Pi := \{(x^j, I^j)\}_{1 \le j \le m}$, $x^j \in I^j$
  $I^j \subset I$ *non-overlapping n-intervals,* $\cup_{j=1}^m I^j = I$
- **gauge** on I : $\delta : I \to \mathbb{R}_+$; Π **δ-fine** *:* $\forall\, j : I^j \subset B[x^j, \delta(x^j)]$
- $f : I \to \mathbb{R}$, **Riemann sum** *:* $S(f, \Pi) := \sum_{j=1}^m f(x^j)|I^j|$
- f **KH-integrable** on I : $\exists\, J \in \mathbb{R}, \forall\, \varepsilon > 0$, $\exists$ *gauge* δ *on* I,
  $\forall\, \delta-$*fine* Π : $|J - S(f, \Pi)| \le \varepsilon$
- $J = \int_I f$ is the **KH-integral** of f on I
- $E \subset I$ **measurable** if 1_E KH-integrable on I, $\mu(E) := \int_I 1_E$
- *Fubini, monotone and dominated convergence thms*
- *no change of variables thm, restriction to finite union of n-intervals*
- f **L-integrable** on I *:* f *and* $|f|$ KH-integrable on I

n-dim. fundamental thm of calculus

- $v \in C^1(A, \mathbb{R}^n),\ A \subset \mathbb{R}^n,\ \partial A$ 'nice'
  $\int_A div\ v = \int_{\partial A} \langle v, n_A \rangle,\quad n_A$ *outer normal on* $\partial A,\ |n_A| = 1$
- $\exists\, v : I \to \mathbb{R}^n$ *differentiable :* $div\ v$ *not KH-integrable on* I
- mimick proof of fundamental theorem for $n = 1$
- $\forall\, \varepsilon > 0,\ \forall\, x \in I,\ \exists\, \delta(x) > 0,\ \forall\, y \in B[x, \delta(x)] :$
  $\|v(y) - v(x) - v'(x)(y - x)\| \leq \varepsilon^2 \|y - x\|$
- $\forall\, x \in I, w_x := v(x) + v'(x)(\cdot - x) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$
- $\Pi = \{(x^j, I^j)\}_{1 \leq j \leq m}\ \delta$-fine $\Rightarrow$
  $\int_{\partial I^j} \langle w_{x^j}, n_{I^j} \rangle = \int_{I^j} div\ w_{x^j} = div\ v(x^j)|I^j|$
  $\int_{\partial I^j} \langle v, n_{I^j} \rangle - div\ v(x^j)|I^j| = \int_{\partial I^j} \langle v - w_{x^j}, n_{I^j} \rangle$
  $\|v(y) - w_{x^j}(y)\| \leq \varepsilon^2 \|y - x^j\|\ \ \forall\, y \in I^j,\ \forall\, j = 1, \ldots, m$

n-dim. fundamental thm of calculus

- $\| \int_{\partial I} \langle v, n_I \rangle - S(div\ v, \Pi) \| \leq \sum_{j=1}^{m} \| \int_{\partial I^j} \langle w_{x^j} - v, n_{I_j} \rangle \|$
  $\leq \varepsilon^2 \sum_{j=1}^{m} d(I^j)\, |\partial I^j| := \varepsilon^2 \sigma(\Pi)$
  $\sigma(\Pi)$ **irregularity** *of* Π, $d(I^j)$ **diameter** *of* I^j
  $|\partial I^j|$ (n-1)-dimensional measure of ∂I^j
- $\| \int_{\partial I} \langle v, n_I \rangle - S(div\ v, \Pi) \| \leq \varepsilon$ if one adds to Π δ-fine the *irregularity restriction* $\sigma(\Pi) \leq \varepsilon^{-1}$
- geometrical meaning : $I^j = I_1^j \times \ldots \times I_n^j$,
  $d(I^j) = \max_{1 \leq k \leq n} |I_k^j|$, $|\partial I^j| \leq \frac{2n|I^j|}{\min_{1 \leq k \leq n} |I_k^j|}$
  $\sigma(\Pi) \leq 2n \max_{1 \leq j \leq m} \frac{\max_{1 \leq k \leq n} |I_k^j|}{\min_{1 \leq k \leq n} |I_k^j|} |I| := 2n\, \sigma_0(\Pi)\, |I|$
- $\| \int_{\partial I} \langle v, n_I \rangle - S(div\ v, \Pi) \| \leq \varepsilon$ if Π satisfies the **stronger irregularity** restriction : $\sigma_0(\Pi) \leq \frac{1}{2n\varepsilon|I|}$

Generalized KH-integrals on n-interval

- $f : I \subset \mathbb{R}^n \to \mathbb{R}, \quad I \quad n$*-interval*
- 1981, M. : **M-integrable** *on* I *if* $\exists\, J \in \mathbb{R}, \forall\, \varepsilon > 0, \exists$ *gauge* δ *on* $I, \forall\, \delta$*-fine* $\Pi, \sigma_0(\Pi) \leq \frac{1}{2n\varepsilon|I|} : |S(f,\Pi) - J| \leq \varepsilon$
- 1983, JARNIK, KURZWEIL, SCHWABIK : **M$_1$-integrable** *on* I *:* replace $\sigma_0(\Pi) \leq \frac{1}{2n\varepsilon|I|}$ by $\sigma(\Pi) \leq \varepsilon^{-1}$
- 1986, PFEFFER *:* **Pf-integrable** *on* I , using *irregularity with respect to a finite family of planes parallel to the coordinate axes*
- 1992, JARNIK, KURZWEIL : **ext-integrable** *on* I if f *extended by* 0 *on some n-interval* $L \supset int\ L \supset I$ *is M-integrable on* L
- *M*$_1$*-int* $\Rightarrow$ *Pf-int* $\Leftrightarrow$ *ext-int* $\Rightarrow$ *M-int*
- all properties of KH-integral except Fubini's thm; divergence thm for differentiable vector field; no change of variable thm

Generalized KH-integrals on $M \subset \mathbb{R}^n$

- $f : M \subset \mathbb{R}^n \to \mathbb{R}$, M *compact*
- 1985, 1988, JARNIK, KURZWEIL : **PU-integral** *on* M *,* **PU-partition** *defined from a suitable partition of unity, irregularity modelled on* σ
- 1991, PFEFFER : **v-integral** *on BV-set* M *,* v *continuous outside of a set of (n-1)-Hausdorff measure zero and almost differentiable outside a set of* σ*-finite (n-1)-Hausdorff measure*
- 1991, KURZWEIL, M., PFEFFER : **G-integral** *on BV-set* M *, BV partitions of unity; same divergence thm*
- 2001, PFEFFER : **R-integral** *on BV-set* M *,* based on **charges**
- 2004, DE PAUW, PFEFFER : *apply R-integral to obtain removable sets of singularities of elliptic equations*
- *other results by* JURKAT, NONNENMACHER, BUCZOLICH, PLOTNIKOV, FLEISCHER, KUNCOVÁ, MALÝ, MOONENS. . .

Thank you for your patience !

More **details** and **references** in

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- TH. DE PAUW, *Autour du théorème de la divergence,* Panorama et synthèses 18 (2004), 85-121
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