Removable singularities for the equation $\operatorname{div} v = f$

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Overview

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Removable sets

Let $\mathscr{B}$ be a collection of measurable vector fields $v : \mathbb{R}^n \to \mathbb{R}^n$ for which the distributional divergence

$$\operatorname{div} v : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}, \varphi \mapsto -\int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx$$

is well-defined.

Definition

A compact set $S \subseteq \mathbb{R}^n$ is said to be $\mathscr{B}$-removable for the equation $\operatorname{div} v = 0$ iff for every $v \in \mathscr{B}$, the equality

$$\langle \operatorname{div} v, \varphi \rangle = 0 \text{ for any } \varphi \in \mathscr{D}(\mathbb{R}^n) \text{ with } \operatorname{supp} \varphi \cap S = \emptyset$$

implies that $\operatorname{div} v \equiv 0$ (in $\mathbb{R}^n$).

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Removable sets (II)

Given a collection $\mathscr{F}$ of measurable functions such that $\nabla\mathscr{F} \subseteq \mathscr{B}$ and using the equality

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A compact set $S \subseteq \mathbb{R}^n$ is said to be $\mathscr{F}$-removable for the equation $\Delta v = 0$ iff for every $f \in \mathscr{F}$,

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A useful lemma

The following lemma is easy to obtain by smoothing the characteristic function of suitable neighborhoods of S.

Lemma (De Pauw, 2000)

Given a compact set $S \subseteq \mathbb{R}^n$ with $\mathscr{H}^{n-1}(S) < +\infty$, there exists a sequence $(\chi_k) \subseteq \mathscr{D}(\mathbb{R}^n)$ satisfying $0 \le \chi_k \le 1$ for each $k \in \mathbb{N}$, together with the following properties :

- $\chi_k = 1$ in a neighborhood of S for each $k \in \mathbb{N}$;
- $\int_{\mathbb{R}^n} \chi_k \, dx \to 0, \, k \to \infty$;
- $\overline{\lim}_{k\to\infty} \int_{\mathbb{R}^n} |\nabla \chi_k| \, dx \le C(n)\mathscr{H}^{n-1}(E)$.

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Overview

1. Definition and preliminaries
2. L^∞-removable sets
   - A sufficient condition for a set to be removable
   - A necessary condition for a set to be removable
   - Comparison with the Laplace equation
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   - A sufficient condition for a set to be removable
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   - A necessary condition for a set to be removable
4. A perspective

A sufficient condition to be L^∞-removable

Observation (De Pauw, 2000)

If the compact $S \subseteq \mathbb{R}^n$ satisfies $\mathscr{H}^{n-1}(S) = 0$, then it is $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$-removable for the equation $\operatorname{div} v = 0$.

Sketch of the proof. Assume that $\mathscr{H}^{n-1}(S) = 0$, fix $v \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and suppose that $\operatorname{div} v = 0$ outside S.

If (χ_k) is the sequence associated to S by the previous lemma, then we get for $\varphi \in \mathscr{D}(\mathbb{R}^n)$:

$$\begin{aligned}\langle \operatorname{div} v, \varphi \rangle &= -\int_{\mathbb{R}^n} v \cdot \nabla(\chi_k \varphi)\, dx \\ &\leq \|v\|_\infty [\|\nabla\varphi\|_\infty \|\chi_k\|_1 + \|\varphi\|_\infty \|\nabla\chi_k\|_1] \to 0, k \to \infty,\end{aligned}$$

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Proposition (M., 2006)

If the compact set $S \subseteq \mathbb{R}^n$ is $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$-removable for $\operatorname{div} v = 0$, then $\mathscr{H}^{n-1}(S) = 0$.

Sketch of the proof (Phuc-Torres, 2009).

- Assume that $0 < \mathscr{H}^{n-1}(S) < +\infty$.
- Choose a (nontrivial) Radon measure μ supported in S satisfying $\mu(B[x,r]) \leq Mr^{n-1}$ (Frostman).
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Last step of the proof :

- observe that the growth condition on μ guarantees that one has

$$\int_{\mathbb{R}^n} \varphi \, d\mu \le C(n,M) \|\nabla \varphi\|_{L^1}$$

  for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$.
- If we let

$$X := \{u \in L^{n/(n-1)}(\mathbb{R}^n) : \nabla u \in L^1(\mathbb{R}^n, \mathbb{R}^n)\}$$

  be endowed by the norm $\|u\|_X := \|\nabla u\|_1$, this yields $\mu \in X^*$.
- Yet $T : X \to L^1, u \mapsto -\nabla u$ is injective.
- So T^* is surjective and $\mu = T^*(g)$ for some $g \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$.
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Comparison with the Laplace equation.

Theorem (David-Mattila, 1999)

If the compact set $S \subseteq \mathbb{R}^2$ satisfying $0 < \mathscr{H}^1(S) < +\infty$ is purely non 1-rectifiable, then S is $\mathrm{Lip}\,(\mathbb{R}^2)$-removable for the Laplace equation.

Theorem (Nazarov-Tolsa-Volberg, 2012 for $n > 2$)

If the compact set $S \subseteq \mathbb{R}^n$ satisfying $0 < \mathscr{H}^{n-1}(S) < +\infty$ is purely non $(n-1)$-rectifiable, then S is $\mathrm{Lip}\,(\mathbb{R}^n)$-removable for the Laplace equation.

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Overview

A sufficient condition to be C^0-removable

Observation (Pfeffer)

Given $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$, let $F := \operatorname{div} v$. For all $\varepsilon > 0$ and all compact set $K \subseteq \mathbb{R}^n$, there exists $\theta > 0$ such that

$$F(\varphi) \leq \theta \|\varphi\|_1 + \varepsilon \|\nabla \varphi\|_1$$

holds for any $\varphi \in \mathscr{D}_K(\mathbb{R}^n)$.

- In fact (De Pauw-Pfeffer, 2008), the above condition characterizes all distributions $F \in \mathscr{D}(\mathbb{R}^n)^*$ which *are* the distributional divergence of some $v \in C^0(\mathbb{R}^n, \mathbb{R}^n)$.
- Hence if $S \subseteq \mathbb{R}^n$ is compact and satisfies $\mathscr{H}^{n-1}(S) < +\infty$, then it is $C^0(\mathbb{R}^n, \mathbb{R}^n)$-removable for $\operatorname{div} v = 0$.

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Comparison with the Laplace equation

Proposition (de Valeriola-M., 2010)

(A) if $S \subseteq \mathbb{R}^n$ is compact and if $\mathscr{H}^{n-1} \llcorner S$ is σ-finite, then S is $C^0(\mathbb{R}^n, \mathbb{R}^n)$-removable for the equation $\operatorname{div} v = 0$;
(Note : latent in De Pauw-Pfeffer, 2003)

(B) there exists a compact set $S \subseteq \mathbb{R}^n$ such that :

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A necessary condition for a set to be removable

Theorem (Ponce, 2012)

A compact set $S \subseteq \mathbb{R}^n$ is $C^0(\mathbb{R}^n, \mathbb{R}^n)$-removable for the equation $\operatorname{div} v = 0$ if and only if $\mathscr{H}^{n-1} \llcorner S$ is σ-finite.

Overview

A perspective

- Weighted Lebesgue spaces : natural framework to solve $\mathrm{div}\, v = 0$ on non-smooth open domains (see *e.g.* Duran-Russ-Tchamitchian, 2010).
- In w-weighted lebesgue space : analogous sufficient and necessary removability conditions where $r \mapsto r^{n-1}$ in the definition of Hausdorff measure is replaced by

$$P_w(B(x,r)) \quad \text{and} \quad \frac{1}{r}\int_{B[x,r]} w\, dx$$

(M.-Russ, 201X).
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