Maximal lineability of the set of continuous surjections

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Sumary

1. Brief overview throughout history
   - Unexpected objects
   - Lineability and surjective functions

2. Does there exist a continuous surjection from $\mathbb{R}^m$ onto $\mathbb{R}^n$?
   - A CS from $\mathbb{R}$ onto $\mathbb{R}^2$
   - A CS from $\mathbb{R}^m$ onto $\mathbb{R}^n$

3. $S_{m,n}$ lineability
   - A family of CS functions
   - Main result

4. References
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Unexpected objects...

In 1872, K. Weierstrass provided the classical example of a function that was continuous everywhere but differentiable nowhere:

\[ f(x) = \sum_{n=0}^{\infty} a_n \cos(bn\pi x) \]

where \( 0 < a < 1 \), \( b \) is an odd integer and \( ab > 1 + \frac{3\pi}{2} \). "Weierstrass' monsters" were also found by B. Bolzano (1830), M. Ch. Cellérier (1830), B. Riemann (1861) and H. Hankel (1870).
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Brief overview throughout history

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Theorem (Gurariy, 1966)

*The set of continuous nowhere differentiable functions on \([0, 1]\) contains, except for the zero function, an infinite linear space.*
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Let $X$ be a topological vector space, $M$ a subset of $X$ and $\mu$ a cardinal number. $M$ is said to be $\mu$-lineable ($\mu$-spaceable) if $M \cup \{0\}$ contains a vector space (a closed vector space) $\mu$-dimensional.
Everywhere surjective functions
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Definition

An everywhere surjective (ES) function $f \in \mathbb{R}^\mathbb{R}$ satisfies $f(I) = \mathbb{R}$, for every non degenerated interval $I \subset \mathbb{R}$. 
## Everywhere surjective functions

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### Theorem (Aron, Gurariy and Seoane-Sepúlveda, 2005)

*The space $ES(\mathbb{R})$ of ES maps is maximal lineable in $\mathbb{R}^\mathbb{R}$.*
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An everywhere surjective (ES) function \( f \in \mathbb{R}^\mathbb{R} \) satisfies \( f(I) = \mathbb{R} \), for every non degenerated interval \( I \subset \mathbb{R} \).

Theorem (Aron, Gurariy and Seoane-Sepúlveda, 2005)

The space \( \text{ES}(\mathbb{R}) \) of ES maps is maximal lineable in \( \mathbb{R}^\mathbb{R} \).

Indeed, other classes of functions with even worse behaviour are as large as the ES ones

Theorem (J. L. Gámez-Merino, 2011)

The space \( \text{J}(\mathbb{R}) \) of Jones functions is maximal lineable in \( \mathbb{R}^\mathbb{R} \).
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add \textit{continuity condition} condition and

and ask about \textit{lineability} in this situation.
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   - A family of CS functions
   - Main result

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Let $m$ and $n$ be positive integers. Throughout this we shall denote

$$S_{m,n} = \{ f : \mathbb{R}^m \to \mathbb{R}^n ; f \text{ is continuous and surjective} \}.$$
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$$\mathcal{S}_{m,n} = \{ f : \mathbb{R}^m \rightarrow \mathbb{R}^n ; f \text{ is continuous and surjective} \}.$$ 

But, is $\mathcal{S}_{m,n}$ nonempty?
Does there exist a continuous surjection from $\mathbb{R}^m$ onto $\mathbb{R}^n$?

A CS from $\mathbb{R}$ onto $\mathbb{R}^2$

Peano Curves
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Peano Curves

We use the fact that exists a Peano Curve or a Scale Filling Curve on the square $I^2$ (here $I$ denotes the closed interval $[0, 1]$):
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**Theorem (G. Peano, 1890)**

*There exists a continuous surjection from $I$ to $I^2$.***
Peano Curves

We use the fact that exists a *Peano Curve* or a *Scale Filling Curve* on the square $I^2$ (here $I$ denotes the closed interval $[0, 1]$):

**Theorem (G. Peano, 1890)**

*There exists a continuous surjection from $I$ to $I^2$.***

**Theorem (A.D. Alexandrov)**

*There is a continuous surjection from the Cantor space $\mathcal{K}$ onto any arbitrary nonempty compact metric space.*
A geometric construction of a space filling curve...
A continuous surjection from \( \mathbb{R} \) onto \( \mathbb{R}^2 \)

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1. Write $\mathbb{R}^2 = \bigcup_{n \in \mathbb{Z}} Q_n$, where $Q_n$ denote the closed unit squares with the inferior left corner in $\mathbb{Z}^2$. 
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One may to construct a CS $\mathbb{R} \to \mathbb{R}^2$ following these steps:

1. Write $\mathbb{R}^2 = \bigcup_{n \in \mathbb{Z}} Q_n$, where $Q_n$ denote the closed unit squares with the inferior left corner in $\mathbb{Z}^2$.

2. Built continuous surjections

$$F_n : \left[ n, n + \frac{1}{2} \right] \to Q_n.$$ 

for each integer $n$. Thus, we may define continuous maps $G_n : [n + \frac{1}{2}, n + 1] \to \mathbb{R}^2$ such that starts/ends at the end/initial point of the curve $F_n/F_{n+1}$, respectively.
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3. Pasting the functions $F_n$ and $G_n$, we obtain a CS $\mathbb{R} \to \mathbb{R}^2$. 


A continuous surjection from $\mathbb{R}^m$ onto $\mathbb{R}^n$

From a CS $F : \mathbb{R} \to \mathbb{R}^2$...
Does there exist a continuous surjection from $\mathbb{R}^m$ onto $\mathbb{R}^n$?

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...and thus a CS $\mathbb{R}^m \rightarrow \mathbb{R}^n$. 
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A family of CS functions

Inspired on results from the R. Aron's, V.I. Gurariy's, and J.B. Seoane-Sepúlveda's paper 'Lineability and spaceability of sets of functions on \( \mathbb{R} \), Proc. Amer. Math. Soc. (2005), we define, for each positive real \( r \in \mathbb{R}^+ \), the homeomorphism \( \phi_r : \mathbb{R} \to \mathbb{R} \) by

\[
\phi_r(t) := e^{rt} - e^{-rt}.
\]

Thus, the subset \( A := \{ \phi_r \} \) of \( \mathbb{R} \) is linearly independent, has cardinality \( c \), and every nonzero element of span \( (A) \) is continuous and surjective.
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Thus,

**Lemma**

*The subset \( \mathcal{A} := \{\phi_r\}_{r \in \mathbb{R}^+} \) of \( \mathbb{R}^\mathbb{R} \) is linearly independent, has cardinality \( c \), and every nonzero element of \( \text{span}(\mathcal{A}) \) is continuous and surjective.*
Indeed, we may suppose $r_1 > r_2 > \ldots > r_k > 0$ and, then, write

$$
\left( \sum_{i=1}^{k} \alpha_i \cdot \phi_{r_i} \right) (t) = e^{r_1 t} \cdot \left( \alpha_1 + \sum_{i=2}^{k} \alpha_i \cdot e^{(r_i-r_1)t} \right) - \sum_{i=1}^{k} \alpha_i \cdot e^{-r_i t}
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\]

Now, for each \( r = (r_1, \ldots, r_n) \in (\mathbb{R}^+)^n \), let \( \varphi_r : \mathbb{R}^n \to \mathbb{R}^n \) be the homeomorphism defined by \( \varphi_r = (\phi_{r_1}, \ldots, \phi_{r_n}) \), i.e.,

\[
\varphi_r(x) := (\phi_{r_1}(x_1), \ldots, \phi_{r_n}(x_n)),
\]

for all \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).
Working on each coordinate, and using the previous lemma, we have the following.

**Lemma**

The set $\mathcal{B} = \{\varphi_r\}_{r \in (\mathbb{R}^+)^n}$ of $\mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$ is linearly independent, has cardinality $\mathfrak{c}$, and every nonzero element of span($\mathcal{B}$) is continuous and surjective.
Main result

Theorem (1)

$S_{m,n}$ is $c$-lineable and, therefore, maximal lineable in $C(\mathbb{R}^m, \mathbb{R}^n)$. 

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Sketch of the proof: Let’s fix $F \in S_{m,n}$.

\textsuperscript{1}to appear in Bull. Belg. Math. Soc. Simon Stevin
Main result

Theorem (1)

\( S_{m,n} \) is \( \mathcal{C} \)-lineable and, therefore, maximal lineable in \( C(\mathbb{R}^m, \mathbb{R}^n) \).

**Sketch of the proof:** Let’s fix \( F \in S_{m,n} \).

Using the notation of the previous lemma, we will prove that

\[ \mathcal{C} = \{ \Phi \circ F \}_{\Phi \in \mathcal{B}} \]

is such that \( \text{span}(\mathcal{C}) \) is the space we are looking for.

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Main result

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$S_{m,n}$ is $c$-lineable and, therefore, maximal lineable in $C(\mathbb{R}^m, \mathbb{R}^n)$.

Sketch of the proof: Let’s fix $F \in S_{m,n}$.

Using the notation of the previous lemma, we will prove that

$$\mathcal{C} = \{\Phi \circ F\}_{\Phi \in \mathcal{B}}$$

is such that span$(\mathcal{C})$ is the space we are looking for.

The surjectivity of $F$ assures that $G \circ F = 0$ implies $G = 0$, for every function $G : \mathbb{R}^n \to \mathbb{R}^n$.

So, Thus, if $\Phi_i \in \mathcal{V}$, $i = 1, \ldots, k$ and

$$0 = \sum_{i=1}^{k} \alpha_i \cdot \Phi_i \circ F = \left( \sum_{i=1}^{k} \alpha_i \Phi_i \right) \circ F,$$

then $\alpha_i = 0$, $i = 1, \ldots, k$ and, thus, $\mathcal{C}$ is linearly independent.
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then $\alpha_i = 0$, $i = 1, \ldots, k$ and, thus, $\mathcal{C}$ is linearly independent.

Furthermore, any nonzero function

$$\sum_{i=1}^{l} \lambda_i \cdot \Psi_i \circ F = \left( \sum_{i=1}^{l} \lambda_i \Psi_i \right) \circ F$$

of $\text{span}(\mathcal{C})$ is continuous and surjective.
Maximal lineability of the set of continuous surjections

\( S_{m,n} \) lineability

Main result

...and this in higher dimension?
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Let’s denote the space of all real sequences by

\[ \mathbb{R}^N = \mathbb{R} \times \mathbb{R} \times \cdots \]

and equip it with the product topology.
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And about \( S_{n,\mathbb{N}} \) lineability?
...and this in higher dimension?

Let’s denote the space of all real sequences by

\[ \mathbb{R}^N = \mathbb{R} \times \mathbb{R} \times \cdots \]

and equip it with the product topology.

And about \( S_{n,N} \) lineability?

There is nothing to be done, since

“there is no CS map \( \mathbb{R} \rightarrow \mathbb{R}^N \)”

implies \( S_{n,N} = \emptyset \).
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Thank you very much for your attention!