Maximal lineability of the set of continuous surjections

Nacib Gurgel Albuquerque

Federal University of Paraíba, Brazil

The Sugarcane Symposium in Real Analysis XXXVII
ICMC USP–São Carlos
6th June 2013

Sumary

Sumary

Unexpected objects...

In 1872, K. Weierstrass provided the classical example of a function that was continuous everywhere but differentiable nowhere:

Unexpected objects...

In 1872, K. Weierstrass provided the classical example of a function that was continuous everywhere but differentiable nowhere:

$$f(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is an odd integer and $ab > 1 + 3\pi/2$.

Unexpected objects...

In 1872, K. Weierstrass provided the classical example of a function that was continuous everywhere but differentiable nowhere:

$$f(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is an odd integer and $ab > 1 + 3\pi/2$.

"Weierstrass' monsters" were also found by B. Bolzano (1830), M.Ch. Cellérier (1830), B. Riemann (1861) and H. Hankel (1870).

...appears more often than we might think?

...appears more often than we might think?

Theorem (Gurariy, 1966)

The set of continuous nowhere differentiable functions on $[0, 1]$ *contains, except for the zero function, a infinite linear space.*

...appears more often than we might think?

Theorem (Gurariy, 1966)

The set of continuous nowhere differentiable functions on $[0, 1]$ contains, except for the zero function, a infinite linear space.

Definition (Aron, Gurariy, Seoane–Sepúlveda, 2004)

Let X be a topological vector space, M a subset of X and μ a cardinal number. M is said to be μ-lineable (μ-spaceable) if $M \cup \{0\}$ contains a vector space (a closed vector space) μ-dimensional.

Everywhere surjective functions

Everywhere surjective functions

Definition

An everywhere surjective (ES) function $f \in \mathbb{R}^{\mathbb{R}}$ satisfies $f(I) = \mathbb{R}$, for every non degenerated interval $I \subset \mathbb{R}$.

Everywhere surjective functions

Definition

An everywhere surjective (ES) function $f \in \mathbb{R}^{\mathbb{R}}$ satisfies $f(I) = \mathbb{R}$, for every non degenerated interval $I \subset \mathbb{R}$.

Theorem (Aron, Gurariy and Seoane-Sepúlveda, 2005)

The space $ES(\mathbb{R})$ of ES maps is maximal lineable in $\mathbb{R}^{\mathbb{R}}$.

Everywhere surjective functions

Definition

An everywhere surjective (ES) function $f \in \mathbb{R}^{\mathbb{R}}$ satisfies $f(I) = \mathbb{R}$, for every non degenerated interval $I \subset \mathbb{R}$.

Theorem (Aron, Gurariy and Seoane-Sepúlveda, 2005)

The space $ES(\mathbb{R})$ of ES maps is maximal lineable in $\mathbb{R}^{\mathbb{R}}$.

Indeed, other classes of functions with even worse behaviour are as large as the ES ones

Theorem (J. L. Gámez-Merino, 2011)

The space $J(\mathbb{R})$ of Jones functions is maximal lineable in $\mathbb{R}^{\mathbb{R}}$.

these previous classes mentioned are very exotic, furthermore these are nowhere continuous.

these previous classes mentioned are very exotic, furthermore these are nowhere continuous.

So, a natural question would be

these previous classes mentioned are very exotic, furthermore these are nowhere continuous.

So, a natural question would be

add continuity condition condition and

and ask about lineability in this situation.

Sumary

Let m and n be positive integers. Throughout this we shall denote

$$\mathcal{S}_{m,n} = \{f : \mathbb{R}^m \longrightarrow \mathbb{R}^n \; ; \; f \text{ is continuous and surjective}\}.$$

Let m and n be positive integers. Throughout this we shall denote

$$\mathcal{S}_{m,n} = \{f : \mathbb{R}^m \longrightarrow \mathbb{R}^n \; ; \; f \text{ is continuous and surjective}\}.$$

But, is $\mathcal{S}_{m,n}$ nonempty?

Peano Curves

Peano Curves

We use the fact that exists a *Peano Curve* or a *Scale Filling Curve* on the square I^2 (here I denotes the closed interval $[0, 1]$):

Peano Curves

We use the fact that exists a *Peano Curve* or a *Scale Filling Curve* on the square I^2 (here I denotes the closed interval $[0, 1]$):

Theorem (G. Peano, 1890)

There exists a continuous surjection from I to I^2.

Peano Curves

We use the fact that exists a *Peano Curve* or a *Scale Filling Curve* on the square I^2 (here I denotes the closed interval $[0,1]$):

Theorem (G. Peano, 1890)

There exists a continuous surjection from I to I^2.

Theorem (A.D. Alexandrov)

There is a continuous surjection from the Cantor space $\mathcal{K}$ onto any arbitrary nonempty compact metric space.

Hilbert Curve

A geometric construction of a space filling curve...

A continuous surjection from $\mathbb{R}$ onto $\mathbb{R}^2$

One may to construct a CS $\mathbb{R} \to \mathbb{R}^2$ following these steps:

A continuous surjection from $\mathbb{R}$ onto $\mathbb{R}^2$

One may to construct a CS $\mathbb{R} \to \mathbb{R}^2$ following these steps:

1. Write $\mathbb{R}^2 = \bigcup_{n\in\mathbb{Z}} Q_n$, where Q_n denote the closed unit squares with the inferior left corner in $\mathbb{Z}^2$.

A continuous surjection from $\mathbb{R}$ onto $\mathbb{R}^2$

One may to construct a CS $\mathbb{R} \to \mathbb{R}^2$ following these steps:

1. Write $\mathbb{R}^2 = \bigcup_{n\in\mathbb{Z}} Q_n$, where Q_n denote the closed unit squares with the inferior left corner in $\mathbb{Z}^2$.
2. Built continuous surjections

$$F_n : \left[n, n + \frac{1}{2}\right] \to Q_n.$$

for each integer n. Thus, we may define continuous maps $G_n : [n + \frac{1}{2}, n + 1] \to \mathbb{R}^2$ such that starts/ends at the end/initial point of the curve F_n/F_{n+1}, respectively.

A continuous surjection from $\mathbb{R}$ onto $\mathbb{R}^2$

One may to construct a CS $\mathbb{R} \to \mathbb{R}^2$ following these steps:

1. Write $\mathbb{R}^2 = \bigcup_{n\in\mathbb{Z}} Q_n$, where Q_n denote the closed unit squares with the inferior left corner in $\mathbb{Z}^2$.
2. Built continuous surjections
$$F_n : \left[n, n + \frac{1}{2}\right] \to Q_n.$$
for each integer n. Thus, we may define continuous maps $G_n : [n + \frac{1}{2}, n + 1] \to \mathbb{R}^2$ such that starts/ends at the end/initial point of the curve F_n/F_{n+1}, respectively.
3. Pasting the functions F_n and G_n, we obtain a CS $\mathbb{R} \to \mathbb{R}^2$.

A continuous surjection from $\mathbb{R}^m$ onto $\mathbb{R}^n$

From a CS $F : \mathbb{R} \to \mathbb{R}^2$...

A continuous surjection from $\mathbb{R}^m$ onto $\mathbb{R}^n$

From a CS $F : \mathbb{R} \to \mathbb{R}^2$...

...we get a CS $\mathbb{R} \to \mathbb{R}^n$...

A continuous surjection from $\mathbb{R}^m$ onto $\mathbb{R}^n$

From a CS $F : \mathbb{R} \to \mathbb{R}^2$...

...we get a CS $\mathbb{R} \to \mathbb{R}^n$...

...and thus a CS $\mathbb{R}^m \to \mathbb{R}^n$.

Sumary

A family of CS functions

A family of CS functions

Inspired on results from the R. Aron's, V.I. Gurariy's, and J.B. Seoane-Sepúlveda's paper *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc. (2005), we define, for each positive real $r \in \mathbb{R}^+$, the homeomorphism $\phi_r : \mathbb{R} \to \mathbb{R}$ by

$$\phi_r(t) := e^{rt} - e^{-rt}.$$

A family of CS functions

Inspired on results from the R. Aron's, V.I. Gurariy's, and J.B. Seoane-Sepúlveda's paper *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc. (2005), we define, for each positive real $r \in \mathbb{R}^+$, the homeomorphism $\phi_r : \mathbb{R} \to \mathbb{R}$ by

$$\phi_r(t) := e^{rt} - e^{-rt}.$$

Thus,

Lemma

The subset $\mathfrak{A} := \{\phi_r\}_{r \in \mathbb{R}^+}$ *of* $\mathbb{R}^{\mathbb{R}}$ *is linearly independent, has cardinality* $\mathfrak{c}$*, and every nonzero element of span*$(\mathfrak{A})$ *is continuous and surjective.*

Indeed, we may suppose $r_1 > r_2 > \ldots > r_k > 0$ and, then, write

$$\left(\sum_{i=1}^{k} \alpha_i \cdot \phi_{r_i}\right)(t) = e^{r_1 t} \cdot \left(\alpha_1 + \sum_{i=2}^{k} \alpha_i \cdot e^{(r_i - r_1)t}\right) - \sum_{i=1}^{k} \alpha_i \cdot e^{-r_i t}$$

Indeed, we may suppose $r_1 > r_2 > \ldots > r_k > 0$ and, then, write

$$\left(\sum_{i=1}^{k} \alpha_i \cdot \phi_{r_i}\right)(t) = e^{r_1 t} \cdot \left(\alpha_1 + \sum_{i=2}^{k} \alpha_i \cdot e^{(r_i - r_1)t}\right) - \sum_{i=1}^{k} \alpha_i \cdot e^{-r_i t}$$

Now, for each $r = (r_1, \ldots, r_n) \in (\mathbb{R}^+)^n$, let $\varphi_r : \mathbb{R}^n \to \mathbb{R}^n$ be the homeomorphism defined by $\varphi_r = (\phi_{r_1}, \ldots, \phi_{r_n})$, *i.e.*,

$$\varphi_r(x) := (\phi_{r_1}(x_1), \ldots, \phi_{r_n}(x_n)),$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Working on each coordinate, and using the previous lemma, we have the following.

Lemma

The set $\mathfrak{B} = \{\varphi_r\}_{r \in (\mathbb{R}^+)^n}$ of $\mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$ is linearly independent, has cardinality $\mathfrak{c}$, and every nonzero element of span$(\mathfrak{B})$ is continuous and surjective.

Main result

Theorem ([1])

$\mathcal{S}_{m,n}$ is $\mathfrak{c}$-lineable and, therefore, maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$.

[1] to appear in Bull. Belg. Math. Soc. Simon Stevin

Main result

Theorem ([1])

$\mathcal{S}_{m,n}$ is $\mathfrak{c}$-lineable and, therefore, maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$.

Sketch of the proof: Let's fix $F \in \mathcal{S}_{m,n}$.

[1] to appear in Bull. Belg. Math. Soc. Simon Stevin

Main result

Theorem ([1])

$\mathcal{S}_{m,n}$ is $\mathfrak{c}$-lineable and, therefore, maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$.

Sketch of the proof: Let's fix $F \in \mathcal{S}_{m,n}$.

Using the notation of the previous lemma, we will prove that

$$\mathfrak{C} = \{\Phi \circ F\}_{\Phi \in \mathfrak{B}}$$

is such that $\operatorname{span}(\mathfrak{C})$ is the space we are looking for.

[1] to appear in Bull. Belg. Math. Soc. Simon Stevin

Main result

Theorem ([1])

$\mathcal{S}_{m,n}$ is $\mathfrak{c}$-lineable and, therefore, maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$.

Sketch of the proof: Let's fix $F \in \mathcal{S}_{m,n}$.

Using the notation of the previous lemma, we will prove that

$$\mathfrak{C} = \{\Phi \circ F\}_{\Phi \in \mathfrak{B}}$$

is such that $\text{span}(\mathfrak{C})$ is the space we are looking for.

The surjectivity of F assures that $G \circ F = 0$ implies $G = 0$, for every function $G : \mathbb{R}^n \to \mathbb{R}^n$.

[1] to appear in Bull. Belg. Math. Soc. Simon Stevin

So, Thus, if $\Phi_i \in \mathfrak{B}$, $i = 1, \ldots, k$ and

$$0 = \sum_{i=1}^{k} \alpha_i \cdot \Phi_i \circ F = \left(\sum_{i=1}^{k} \alpha_i \Phi_i \right) \circ F,$$

then $\alpha_i = 0$, $i = 1, \ldots, k$ and, thus, $\mathfrak{C}$ is linearly independent.

So, Thus, if $\Phi_i \in \mathfrak{B}$, $i = 1, \ldots, k$ and

$$0 = \sum_{i=1}^{k} \alpha_i \cdot \Phi_i \circ F = \left(\sum_{i=1}^{k} \alpha_i \Phi_i \right) \circ F,$$

then $\alpha_i = 0$, $i = 1, \ldots, k$ and, thus, $\mathfrak{C}$ is linearly independent.

Furthermore, any nonzero function

$$\sum_{i=1}^{l} \lambda_i \cdot \Psi_i \circ F = \left(\sum_{i=1}^{l} \lambda_i \Psi_i \right) \circ F$$

of $\mathrm{span}(\mathfrak{C})$ is continuous and surjective. □

...and this in higher dimension?

...and this in higher dimension?

Let's denote the space of all real sequences by

$$\mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \cdots$$

and equip it with the product topology.

...and this in higher dimension?

Let's denote the space of all real sequences by

$$\mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \cdots$$

and equip it with the product topology.

And about $\mathcal{S}_{n,\mathbb{N}}$ lineability?

...and this in higher dimension?

Let's denote the space of all real sequences by

$$\mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \cdots$$

and equip it with the product topology.

And about $\mathcal{S}_{n,\mathbb{N}}$ lineability?

There is nothing to be done, since

$$\text{“}\textit{there is no CS map } \mathbb{R} \to \mathbb{R}^{\mathbb{N}}\text{”}$$

implies $\mathcal{S}_{n,\mathbb{N}} = \emptyset$.

Sumary

References

N.G.A., *Maximal lineability of the set of continuous surjections*, Bull. Belg. Math. Soc. Simon Stevin, in press.

R.M. Aron, V.I. Gurariy and J.B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc., **133** (2005), no. 3, 795–803 (electronic).

R.M. Aron, J.B. Seoane-Sepúlveda, *Algebrability of the set of everywhere surjective functions on* $\mathbb{C}$, Bull. Belg. Math. Soc. Simon Stevin, **14** (2007), no. 1, 25–31.

L. Bernal-González, *Algebraic genericity of strict-order integrability*, Studia Math., **199** (2010), no. 3, 279–293.

L. Bernal-González, D.M. Pellegrino, J.B. Seoane-Sepúlveda *Linear subsets of nonlinear sets in topological vector spaces*, Bull. Amer. Math. Soc. (N.S.), in press.

J.L. Gámez-Merino, J. L., *Large algebraic structures inside the set of surjective functions*, Bull. Belg. Math. Soc. Simon Stevin, **18** (2011), no. 2, 297–300.

References

J.L. Gámez-Merino, G.A. Muñoz-Fernández, V.M. Sánchez, J.B. Seoane-Sepúlveda, *Sierpiński-Zygmund functions and other problems on lineability*, Proc. Amer. Math. Soc., **138** (2010), no. 11, 3863–3876.

J. L. Gámez-Merino, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, *Lineability and additivity in* $\mathbb{R}^{\mathbb{R}}$ J. Math. Anal. Appl., **369** (2010), no. 1, 265–272.

A.B. Kharazishvili, *Strange functions in real analysis*, 2nd ed., Pure and Applied Mathematics (Boca Raton), vol. 272, Chapman & Hall/CRC, Boca Raton, FL, 2006.

J.R. Munkres, *Topology*, 2nd ed. Prentice Hall, Upper Saddle River, NJ, 2000.

G. Peano, *Sur une courbe, qui remplit toute une aire plane*, Math. Ann, **36** (1890), no. 1, 157–160.

Thank you very much for your attention!