On some generalization of the classical concept of bounded variation with applications

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Λ-variation

Definition (Waterman, 1972)

Let $f\colon [0,1] \to \mathbb{R}$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of positive real numbers such that $\sum_{i=1}^{\infty} 1/\lambda_i = +\infty$.

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$$\operatorname{var}_\Lambda f = \sup \sum_{i=1}^{n} \frac{1}{\lambda_i} |f(b_i) - f(a_i)|,$$

where the supremum is taken over all finite collections $I_1, \ldots, I_n$ of compact non-overlapping subintervals of $[0,1]$ of the form $I_i := [a_i, b_i]$, is called the Λ-variation of the function f.

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Let $f\colon [0, 2\pi] \to \mathbb{R}$ *be a function of harmonic bounded variation. Then its Fourier series converges to* $\frac{1}{2}[f(t+0) + f(t-0)]$ *at every* $t \in [0, 2\pi]$*, and moreover, converges uniformly to* f *on each interval of the continuity of the function.*

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- Simplification of reasonings.

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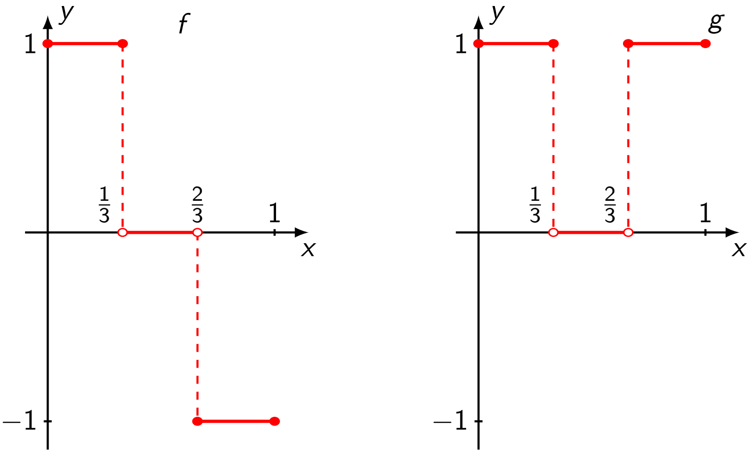
Example 1



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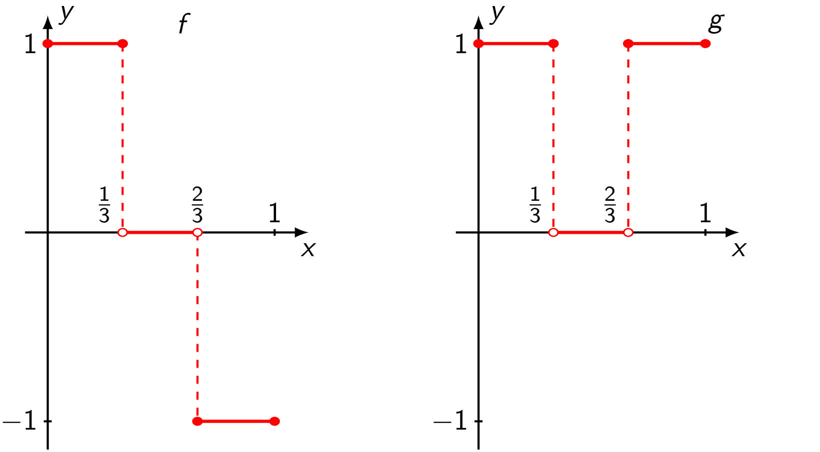

var $f = 2$ and $\mathrm{var}_\Lambda f = 2/\lambda_1$

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Basic properties of ΛBV-functions

- If f is a BV-function, then f is also a ΛBV-function. Moreover,

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- If f is a monotone function, then $\mathrm{var}_\Lambda f = \dfrac{1}{\lambda_1}|f(1) - f(0)|$.
- If f is a ΛBV-function, then f is bounded. Moreover,

$$\sup_{t\in[0,1]} |f(t)| \leq |f(0)| + \lambda_1 \,\mathrm{var}_\Lambda f.$$

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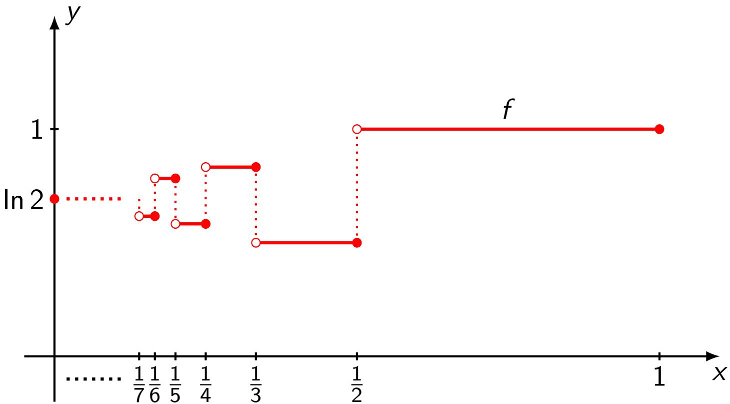
Example 2



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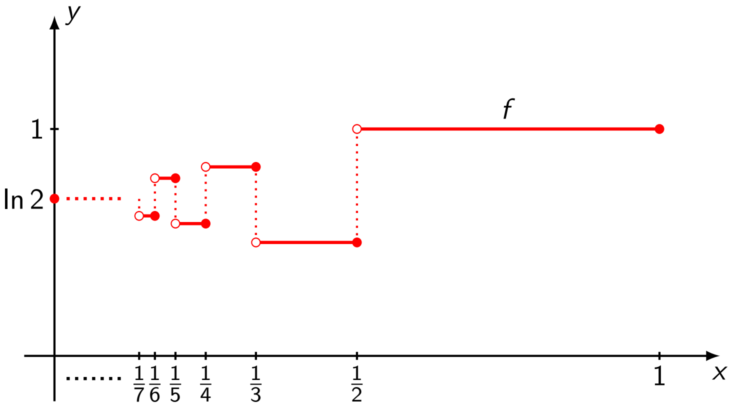

f is not a BV-function, but f is a HBV-function

The space $\Lambda BV[0,1]$

The class of ΛBV-functions, that is,

$$\Lambda BV[0,1] := \{f\colon [0,1] \to \mathbb{R} : \operatorname{var}_\Lambda f < +\infty\},$$

endowed with the norm $\|f\|_\Lambda := |f(0)| + \operatorname{var}_\Lambda f$ is a Banach space.

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- $\Lambda BV[0,1]$ is a Banach algebra in the norm $|f| := \|f\|_\infty + \operatorname{var}_\Lambda f$, since

$$\operatorname{var}_\Lambda(fg) \leq \|g\|_\infty \cdot \operatorname{var}_\Lambda f + \|f\|_\infty \cdot \operatorname{var}_\Lambda g.$$

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- $\Lambda BV[0,1]$ is a Banach algebra in the norm $|f| := \|f\|_\infty + \operatorname{var}_\Lambda f$.
- $\Lambda BV[0,1]$ is not separable; for $\delta \in [0,1]$ consider

$$f_\delta(t) = \begin{cases} 1, & \text{if } t = \delta, \\ 0, & \text{if } t \neq \delta. \end{cases}$$

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- $\Lambda BV[0,1]$ is not separable.
- $\Lambda BV[0,1]$ is not reflexive (see [Prus-Wiśniowski, Ruckle, 2012]).

ΛBV-solutions to Hammerstein integral equation

Consider the nonlinear Hammerstein integral equation

$$x(t) = g(t) + \alpha \int_0^1 k(t,s) f(x(s)) \mathrm{d}s \qquad t \in [0,1], \qquad \text{(H)}$$

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where

- $g \in \Lambda BV[0,1]$;
- $f\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz function;
- $k\colon [0,1] \times [0,1] \to \mathbb{R}$ is a function such that $\operatorname{var}_\Lambda k(\cdot, s) \leq m(s)$ a.e. on $[0,1]$, where $m\colon [0,1] \to [0,+\infty)$ is L-integrable, and for every $t \in [0,1]$ the function $s \mapsto k(t,s)$ is L-integrable.

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Theorem (Bugajewska, O'Regan, 2005)

Under these assumptions there exists $\rho > 0$ such that for every α with $|\alpha| < \rho$ the equation (H) *has a unique solution in* $\Lambda BV[0,1]$.

Lower Λ-variation

Definition

Let $f\colon [0,1] \to \mathbb{R}$ and fix a Λ-sequence $(\lambda_n)_{n\in\mathbb{N}}$. The number

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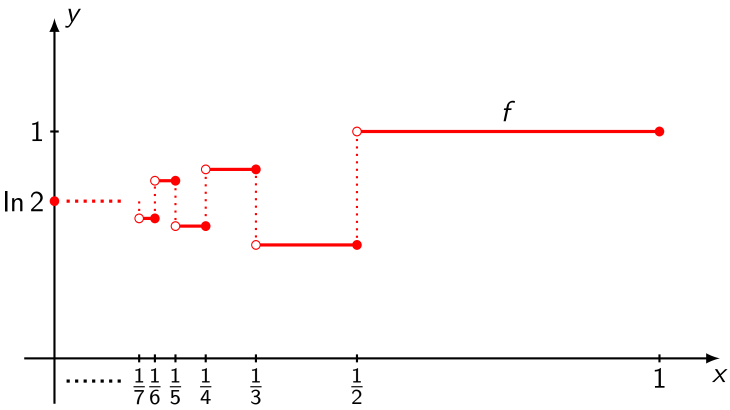
The vector space

$$\underline{\Lambda BV}[0,1] := \left\{f \in L^1[0,1] : \underline{\mathrm{var}}_\Lambda f < +\infty\right\}$$

endowed with the norm $|f|_\Lambda := \|f\|_1 + \underline{\mathrm{var}}_\Lambda f$ is a Banach space.

Example 2, revisited



$f \notin \underline{BV}[0,1]$, but $f \in \underline{HBV}[0,1]$

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- Each $f \in \underline{\Lambda BV}[0,1]$ has a good representative, that is, a function $\varphi \colon [0,1] \to \mathbb{R}$ a.e. equal to f on $[0,1]$ such that $\operatorname{var}_\Lambda \varphi = \underline{\operatorname{var}}_\Lambda f$.

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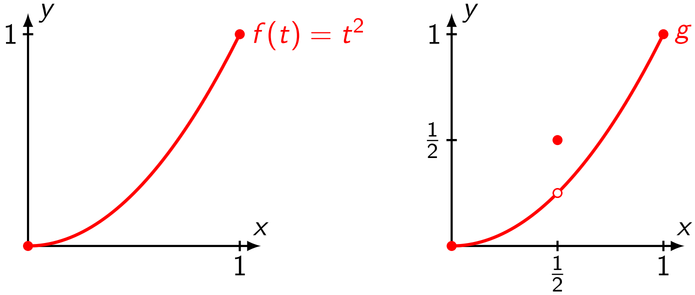
  However, a good representative may not be uniquely determined.



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- A monotone function $f\colon [0,1] \to \mathbb{R}$ is a good representative of $[f] \in \underline{\Lambda BV}[0,1]$, if and only if it is right-continuous at 0 and left-continuous at 1.

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- If $f \in \underline{\Lambda BV}[0,1]$, then $f \in L^\infty[0,1]$. Moreover, there exists a constant c_Λ such that $\|f\|_\infty \leq c_\Lambda \, \mathbf{|} f \mathbf{|}_\Lambda$.

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$$x(t) = g(t) + \alpha \int_0^1 k(t,s) f(x(s)) \mathrm{d}s \qquad \text{for a.e. } t \in [0,1], \quad (\underline{\mathrm{H}})$$

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- there exists a L-measurable function $\vartheta\colon [0,1] \times [0,1] \to \mathbb{R}$ such that $\vartheta(\cdot, s)$ is a good representative of the equivalence class generated by $k(\cdot, s)$ for a.e. $s \in [0,1]$ and for every $t \in [0,1]$ the function $s \mapsto \vartheta(t,s)$ is L-measurable.

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Theorem

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If f is a Lipschitzian, then $(\underline{\mathrm{H}})$ has a unique $\underline{\Lambda BV}$-solution.

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Among all the solutions of $(\underline{\mathrm{H}})$ in $\Lambda BV[0,1]$ 'the best' ones are the good representatives of the unique solution in $\underline{\Lambda BV}[0,1]$.

Thank you
for
your attention

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