Metrizability of Lévy topology on nonadditive measures

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Nonadditive measure

Definition (nonadditive measure)

X: a non-empty set, $\mathcal{A}$: a class of subsets of X with $\emptyset \in \mathcal{A}$. A set function $\mu : \mathcal{A} \to [0, \infty]$ is called a nonadditive measure if it satisfies:

- $\mu(\emptyset) = 0$
- $A, B \in \mathcal{A},\ A \subset B \Rightarrow \mu(A) \leq \mu(B)$ (monotonicity)

It has already appeared in many papers: Hausdorff dimension (Hausdorff 1918), lower/upper numerical probability (Koopman 1940), Maharam's submeasure problem (Maharam 1947), capacity (Choquet 1953/54), semivariation (Dunford-Schwartz 1955), quasimeasure (Alexiuk 1968), maxitive measure (Shilkret 1971), participation measure (Tsichritzis 1971), submeasure (Drewnowski 1972, Dobrakov 1974), fuzzy measure (Sugeno 1974), k-triangular set function (Agafanova-Klimkin 1974), game of characteristic function form, distorted measure (Aumann-Shapley 1974), belief/plausibility function (Shafer 1976), possibility measure (Zadeh 1978), pre-measure (Šipoš 1979), necessity measure (Dubois-Prade 1980), approximately additive (Kalton-Roberts 1983), decomposable measure (Weber 1984), Minkowski-Bouligrand dimension (Schroeder 1991), subjective probabilities in decision making, ...

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The weak convergence and the Lévy convergence on the space of measures

The weak convergence and the Lévy convergence of measures:

- abstract generalizations of the notion of the convergence of distribution functions in probability theory

Let F_n and F be distribution functions and μ_n and μ be the Lebesgue-Stieltjes measures given by F_n and F. The following are equivalent:

1. $F_n(x) \to F(x)$ for every continuity point x of F and $F_n(\infty) \to F(\infty)$
2. $\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu$ for every $f \in C_b(\mathbb{R})$
3. $\mu_n(B) \to \mu(B)$ for every $B \in \mathcal{B}(\mathbb{R})$ with $\mu(\partial B) = 0$

- play an important role when proving many limit theorems in probability theory and statistics, eg: the central limit theorem.

We have two types of generalizations that turn out to be equivalent: a functional analytic one and a measure theoretic one.

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- X: metric space
- $\mathcal{B}(X)$: the σ-field of all Borel subsets of X
- $C_b(X)$: the space of all bounded, continuous real functions on X
- $ca(X)$: the space of all σ-additive Borel measures on X

A functional analytic definition: weak convergence of measures

Let $\{\mu_\alpha\} \subset ca(X)$ be a net and $\mu \in ca(X)$.

$$\mu_\alpha \xrightarrow{w} \mu \overset{def}{\iff} \int_X f d\mu_\alpha \to \int_X f d\mu \text{ for every } f \in C_b(X)$$

- The weak topology on $ca(X)$ is the topology generated by this convergence.
- The weak topology is just the weak* topology on $ca(X)$ generated by the duality

$$(\mu, f) \in ca(X) \times C_b(X) \mapsto \langle \mu, f \rangle := \int_X f d\mu$$

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The portmanteau theorem

A measure theoretic definition: Lévy convergence of measures

$$\mu_\alpha \xrightarrow{L} \mu \overset{def}{\iff} \mu_\alpha(B) \to \mu(B) \text{ for every } \mu\text{-continuity set } B \in \mathcal{B}(X)$$

$B \in \mathcal{B}(X)$ is a μ-continuity set $\overset{def}{\iff} \mu(\partial B) = 0 \iff \mu(B^-) = \mu(B^\circ)$

The portmanteau theorem says that the following are equivalent:

1. $\mu_\alpha \xrightarrow{w} \mu$
2. $\limsup \mu_\alpha(C) \leq \mu(C)$ for every closed C and $\mu_\alpha(X) \to \mu(X)$
3. $\mu(U) \leq \liminf \mu_\alpha(U)$ for every open U and $\mu_\alpha(X) \to \mu(X)$
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In this talk we will:

- introduce a successful analogue of the portmanteau theorem for nonadditive measures, which was given by Girotto and Holzer
- investigate further the possibility of metrizing the weak and Lévy topology on the space of such nonadditive measures.

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Choquet integral

To define weak convergence of nonadditive measures, we will introduce an integral with respect to a nonadditive measure.

additive measure m → Lebesgue integral $\int_X f dm$

nonadditive measure μ → Choquet integral $(C)\int_X f d\mu$

Definition (Choquet integral:1953/54, Schmeidler:1989, K:2008)

Let $f : X \to (-\infty, \infty)$ be a function. The *(asymmetric) Choquet integral* of f with respect to a finite nonadditive μ is defined as:

$$(C)\int_X f d\mu := \int_0^\infty \mu(\{f > t\})dt - \int_{-\infty}^0 \{\mu(X) - \mu(\{f > t\})\}dt$$

$$= (C)\int_X f^+ d\mu - (C)\int_X f^- d\bar{\mu},$$

where $\bar{\mu}(A) := \mu(X) - \mu(A^c)$ is called the *conjugate* of μ.

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It is important to observe:

- The Choquet integral is NOT additive! It is only comonotonically additive.

$$(\mathrm{C})\int_X (f+g)d\mu \neq (\mathrm{C})\int_X f d\mu + (\mathrm{C})\int_X g d\mu \quad \text{unless } f \sim g$$

- The Choquet integral is NOT homogeneous! It is only positively homogeneous.

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The theory of nonadditive measures and Choquet integrals has:

X: finite → A lot of practical applications:
decision models with nonadditive beliefs
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Difficulties when formalizing a nonadditive portmanteau theorem

Now we will introduce a result of Girotto and Holzer (2001).

Among other things we have to answer the following questions:

- What is a reasonable definition of weak convergence of measures?
- What is a proper definition of the μ-continuity set?
- What is an alternative notion of the continuity of measures?

What is a definition of weak convergence of nonadditive measures?

- We expect that a reasonable definition should be given by:

$$\mu_\alpha \xrightarrow{w} \mu \overset{def}{\iff} (\mathrm{C})\int_X f d\mu_\alpha \to (\mathrm{C})\int_X f d\mu \text{ for every } f \in C_b(X)$$

- It will be of interest to study other nonlinear integral cases, for instance, the Sugeno integral.

To answer the 2nd and 3rd questions, we have to carefully investigate some essential problems coming from the nonadditivity of measures!

We can't say more about this story for lack of time, but, in conclusion, in order to give solutions to these questions, Girotto and Holzer introduced the notion of the strong regularity system that we will briefly explain on the next slide.

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Nonadditive version of the μ-continuity set

We begin with defining some regularizations of a nonadditive measure:

Definition (regularity system and strong regularity system)

Let $\mu : \mathcal{B}(X) \to [0, \infty)$ be a nonadditive measure and $A \subset X$.

- outer regularization: $\mu^*(A) := \inf\{\mu(U) : A \subset U, U \text{ is open}\}$
- inner regularization: $\mu_*(A) := \sup\{\mu(C) : C \subset A, C \text{ is closed}\}$
- μ-regularity system:

$$\mathcal{R}_\mu := \{B \in \mathcal{B}(X) : \mu^*(B) = \mu_*(B) = \mu(B)\}$$

- strong outer regularization:

$$\mu^\sharp(A) := \inf\{\mu(C) : A \subset C, C \in \mathcal{R}_\mu \text{ is closed}\}$$

- strong inner regularization:

$$\mu_\sharp(A) := \sup\{\mu(U) : U \subset A, U \in \mathcal{R}_\mu \text{ is open}\}$$

- μ-strong regularity system:

$$\mathcal{R}_\mu^\circ := \{B \in \mathcal{B}(X) : \mu^\sharp(B) = \mu_\sharp(B) = \mu(B)\}$$

Basic properties of μ-strong regularity system

The notion of the strongly regular sets is very useful when formalizing a nonadditive portmanteau theorem.

Proposition

1. $\mathcal{R}_\mu^\circ \subset \mathcal{R}_\mu$ and $\emptyset, X \in \mathcal{R}_\mu^\circ$
2. $B \in \mathcal{R}_\mu \iff B^c \in \mathcal{R}_{\bar{\mu}}$ and $B \in \mathcal{R}_\mu^\circ \iff B^c \in \mathcal{R}_{\bar{\mu}}^\circ$
3. $\mathcal{R}_\mu$ and $\mathcal{R}_\mu^\circ$ are NOT fields!
4. $B \in \mathcal{R}_\mu^\circ \Rightarrow \mu(B^-) = \mu(B^\circ)$
5. Assume that μ is *co-continuous*, i.e.,
   - *c-continuous*: $\mu(C_n) \downarrow \mu(C)$ whenever $\{C_n\}$ is a decreasing sequence of closed sets with $C = \bigcap_{n=1}^\infty C_n$
   - *o-continuous*: $\mu(U_n) \uparrow \mu(U)$ whenever $\{U_n\}$ is an increasing sequence of open sets with $U = \bigcup_{n=1}^\infty U_n$

   Then $B \in \mathcal{R}_\mu^\circ \iff \mu(B^-) = \mu(B^\circ)$.

Due to (4) & (5), $B \in \mathcal{R}_\mu^\circ$ is strong enough to behave as a μ-continuity set in the definition of the Lévy convergence.

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An analogue of the portmanteau theorem for nonadditive measures

We are ready to introduce a nonadditive portmanteau theorem:

Theorem (The nonadditive formalization: Girotto & Holzer 2001)

Let X be a metric space. Let $\{\mu_\alpha\} \subset M(X)$ be a net and $\mu \in M(X)$. Then the following are equivalent:

1. $\mu_\alpha \xrightarrow{w} \mu$
2. $\bar{\mu}_\alpha \xrightarrow{w} \bar{\mu}$
3. *For any closed $C \in \mathcal{R}_\mu$ and any open $U \in \mathcal{R}_\mu$,*

$$\limsup \mu_\alpha(C) \le \mu(C) \quad \text{and} \quad \mu(U) \le \liminf \mu_\alpha(U)$$

4. $\mu_\alpha(B) \to \mu(B)$ *for any* $B \in \mathcal{R}_\mu^\circ$

Definition (Lévy convergence of nonadditive measures)

$$\mu_\alpha \xrightarrow{L} \mu \overset{def}{\iff} \mu_\alpha(B) \to \mu(B) \text{ for every } B \in \mathcal{R}_\mu^\circ$$

This means we can obtain a nonadditive version of the portmanteau theorem if we change the "μ-continuity sets" for the additive case into the "μ-strongly regular sets," which are stronger condition than the "μ-continuity sets," in the definition of the Lévy convergence.

Main Topic I: Metrizing the Lévy topology as a separable space

As is the case for additive measures, to metrize the Lévy topology we need to assume some continuity and regularity conditions:

$$M_{rco}(X) := \left\{ \mu \in M(X) \ : \ \begin{array}{l} \mu \text{ is co-continuous} \\ \mu(B) = \mu^*(B) = \mu_*(B) \text{ for all } B \in \mathcal{B}(X) \end{array} \right\}$$

- It is easily seen that: $M_{rco}(X) = \overline{M_{rco}(X)} := \{\bar{\mu} : \mu \in M_{rco}(X)\}$.
- If μ is autocontinuous and Radon, i.e.,
  - $\mu(A \triangle B_n) \to \mu(A)$ whenever $A, B_n \in \mathcal{B}(X)$ and $\mu(B_n) \to 0$
  - $\forall B \in \mathcal{B}(X)$, $\exists\{K_n\}$: compact sets, $\exists\{U_n\}$: open sets, $K_n \subset B \subset U_n$ and $\mu(U_n \setminus K_n) \to 0$

  then $\mu, \bar{\mu} \in M_{rco}(X)$.

Theorem (Metrizing $M_{rco}(X)$ as a separable space)

Let X be a metric space. Then the following are equivalent:

1. *X is separable*
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In the case of the usual $\mu, \nu \in ca(X)$, we know that:

- Lévy-Prokhorov metric:

$$\rho(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ for all } B \in \mathcal{B}(X)\},$$

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A natural question comes to us:

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Proper nonadditive versions of L-P and F-M metrics

Some difficulties when defining proper nonadditive versions of L-P and F-M metrics:

- $\rho(\mu,\nu) \neq \rho(\nu,\mu)$, i.e., ρ is NOT symmetric!
- $\rho(\mu,\nu) \neq \rho(\bar{\mu},\bar{\nu})$, which means we NEED to calculate $\rho(\bar{\mu},\bar{\nu})$ along with $\rho(\mu,\nu)$ in order to measure the distance between μ and ν!

We expect that proper nonadditive versions of the L-P and F-M metrics should be given by the following formulas:

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Let $\mu,\nu \in M_{rco}(X)$.

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Problem and partial answer

PROBLEM: Is the Lévy topology on $M_{rco}(X)$ metrizable w.r.t. π and κ?

PARTIAL ANSWER: The Lévy topology can be metrized not on the whole space $M_{rco}(X)$ but on a certain subset $\mathcal{P}$ of $M_{rco}(X)$.

Definition (uniform autocontinuity and uniform equi-autocontinuity)

Let $\mu \in M(X)$ and $\mathcal{P} \subset M(X)$.

- μ is *uniformly autocontinuous* $\overset{def}{\iff} \forall\varepsilon > 0, \exists\delta > 0,$ $\forall A, \forall B \in \mathcal{B}(X),$

$$\mu(B) < \delta \ \Rightarrow \ \mu(A \cup B) - \varepsilon < \mu(A) < \mu(A \setminus B) + \varepsilon$$

- $\mathcal{P}$ is *uniformly equi-autocontinuous* $\overset{def}{\iff} \forall\varepsilon > 0, \exists\delta > 0, \forall\mu \in \mathcal{P}, \forall A, \forall B \in \mathcal{B}(X),$

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Examples of uniform equi-autocontinuity sets

Example (uniform equi-autocontinuity set)

- $\mathcal{SUB}(X) := \{\mu \in M(X) : \mu \text{ is subadditive}\}$ is uniformly equi-autocontinuous.
- Let $(\Omega, \mathcal{A})$ be a measurable space and $P : \mathcal{A} \to [0,1]$ be a uniformly autocontinuous nonadditive probability measure. Let $\{\xi_n\}$ be a sequence of X-valued random elements on Ω. Then $\{P \circ \xi_n^{-1}\}$ is uniformly equi-autocontinuous.
- Let $\lambda_1 < 0 < \lambda_2$ be constants. Then

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is uniformly equi-autocontinuous, where μ is said to satisfy *λ-rule* if

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Main theorem

Now we can state our main theorem that gives a partial answer to our problem at this moment:

Theorem

Let $\mathcal{P} \subset M(X)$ be uniformly equi-autocontinuous. Assume that every $\mu \in \mathcal{P}$ is Radon. Then the Lévy topology on $\mathcal{P}$ and $\bar{\mathcal{P}}$ can be metrized w.r.t. π and κ.

The above theorem can be proved by the following uniformity result for weak convergence of measures:

The uniformity for weak convergence on the unit ball in $BL(X, d)$

Let $\{\mu_\alpha\} \subset M(X)$ be uniformly equi-autocontinuous and $\mu \in M(X)$ uniformly autocontinuous. Assume that μ is Radon. The following are equivalent:

1. (C) $\int_X f d\mu_\alpha \to$ (C) $\int_X f d\mu$ for every $f \in BL(X, d)$
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Applications to nonadditive probability theory

Theorem (The nonadditive LeCam theorem)

Let $\{\mu_n\} \subset M(X)$ *and* $\mu \in M(X)$. *Assume that* $\{\mu_n\}$ *is uniformly equi-autocontinuous and every* μ_n *is Radon. If* $\mu_n \xrightarrow{L} \mu$ *and if* μ *is c-continuous and tight, i.e.,* $\forall \varepsilon > 0, \exists K_\varepsilon$*: compact,* $\mu(X \setminus K_\varepsilon) < \varepsilon$*, then* $\{\mu_n\}$ *is uniformly tight, i.e.,*

$$\forall \varepsilon > 0, \exists K_\varepsilon\text{: compact, } \sup_{n\in\mathbb{N}} \mu_n(X \setminus K_\varepsilon) < \varepsilon.$$

Corollary

Let $(\Omega, \mathcal{A})$ *be a measurable space and let* $P : \mathcal{A} \to [0,1]$ *be a uniformly autocontinuous nonadditive probability. Let* ξ *and* ξ_n $(n = 1, 2, \dots)$ *be* X*-valued random elements on* Ω. *Assume that* $P \circ \xi_n^{-1}$ *is Radon and* $P \circ \xi^{-1}$ *is co-continuous and that* $P \circ \xi_n^{-1} \xrightarrow{L} P \circ \xi^{-1}$. *The following are equivalent:*

1. $P \circ \xi^{-1}$ *is tight.*
2. $\{P \circ \xi_n^{-1}\}$ *is uniformly tight.*

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Thank you very much for your attention!