Lyapunov theorems for measure FDEs via Kurzweil-equations

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- Measure FDEs;
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Generalized ODEs

• Let X be a Banach space, $\mathcal{O} \subset X$ an open subset, $\Omega = \mathcal{O} \times [t_0, +\infty)$ and $G : \Omega \to X$.

Definition

We say that $x : [\alpha, \beta] \subset [t_0, +\infty) \to X$ is *a solution of the generalized ODE*

$$\frac{dx}{d\tau} = DG(x, t), \tag{1}$$

in $[\alpha, \beta]$ if $(x(t), t) \in \Omega$ for every $t \in [\alpha, \beta]$ and

$$x(v) - x(\gamma) = \int_{\gamma}^{v} DG(x(\tau), t)$$

$\forall\ \gamma,\, v \in [\alpha, \beta]$.

Measure FDEs

Measure FDEs (MFDEs) are equations of the following form

$$Dx = f(x_t, t)Dg, \tag{2}$$

where x_t is given by $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$, with $r > 0$, $f : G([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$, with $t_0 \geq 0$, Dx and Dg are the distributional derivatives with respect to x and g, in the sense of distributions of L. Schwartz.

When $g(t) = t$, the equation (2) becomes a functional differential equation in the usual sense.

Integral form of a MFDE

The integral form equivalent to (2) is given by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\, dg(s), \quad t \in [t_0, +\infty),$$

where we consider the Kurzweil-Stieljtes integral taken with respect to the nondecreasing function $g : [t_0, +\infty) \to \mathbb{R}$.

The integral $\int_a^b f(t)dg(t)$ is a particular case of the Kurzweil integral $\int_a^b DU(\tau, t)$, when $U : [a, b] \times [a, b] \to X$ is given by $U(\tau, t) = f(\tau)g(t)$.

Relation between equations

We say that a set $O \subset BG\left([t_0 - r, +\infty), \mathbb{R}^n\right)$ has the prolongation property, if for every $y \in O$ and $\bar{t} \in [t_0 - r, +\infty)$, the function $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \leq t \leq \bar{t}, \\ y(\bar{t}), & \bar{t} < t < \infty \end{cases}$$

also belongs to O. Let $S = \{y_t;\ y \in O,\ t \in [t_0, \infty)\}$.

We consider the following MFDE with perturbations

$$Dy = f\left(y_t, t\right) Dg + p(t) Du. \tag{3}$$

where the functions $g, u : [t_0, +\infty) \to \mathbb{R}$ are nondecreasing, $f : S \times [t_0, +\infty) \to \mathbb{R}^n$ and $p : [t_0, \infty) \to \mathbb{R}^n$. Its integral form is given by

$$y\left(t\right) = y(t_0) + \int_{t_0}^{t} f\left(y_s, s\right) dg(s) + \int_{t_0}^{t} p(s) du(s), \quad t \in [t_0, \infty),$$

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where the integrals are considered in the Kurzweil-Stieltjes's sense. We consider the following hypotheses on the functions f and p:

(H_1) The Kurzweil-Stieltjes integral $\int_{t_0}^{t} f(y_s, s)dg(s)$ exists for every $y \in O$ and $t \in [t_0, \infty)$.

(H_2) There exists a function $M : [t_0, \infty) \to \mathbb{R}$ locally Lebesgue-Stieltjes integrable with respect to g such that

$$\left| \int_u^v f(y_s, s)dg(s) \right| \leq \int_u^v M(s)dg(s)$$

$\forall\ y \in O$ and $u, v \in [t_0, \infty)$

(H_3) There exists a function $L : [t_0, \infty) \to \mathbb{R}$ locally Lebesgue-Stieltjes integrable with respect to g such that

$$\left| \int_u^v [f(y_s, s) - f(z_s, s)]dg(s) \right| \leq \int_u^v L(s)\|y_s - z_s\|_\infty dg(s)$$

$\forall\ y, z \in O$ and $u, v \in [t_0, \infty)$.

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$\forall\ y, z \in O$ and $u, v \in [t_0, \infty)$.

(H_4) The Kurzweil-Stieltjes integral $\int_{t_0}^{t} p(s)du(s)$ exists for every $t \in [t_0, \infty)$;

(H_5) There exists a function $K : [t_0, \infty) \to \mathbb{R}$ locally Lebesgue-Stieltjes integrable with respect to u such that, for every $w, v \in [t_0, \infty)$, we have

$$\left| \int_{w}^{v} p(s)du(s) \right| \leq \int_{w}^{v} K(s)du(s).$$

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$$\left| \int_{w}^{v} p(s)du(s) \right| \leq \int_{w}^{v} K(s)du(s).$$

For $y \in O$ and $t \in [t_0, +\infty)$, we define

$$F(y,t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0, \\ \int_{t_0}^{\vartheta} f(y_s, s)dg(s), & t_0 \le \vartheta \le t < +\infty, \\ \int_{t_0}^{t} f(y_s, s)dg(s), & t \le \vartheta < +\infty \end{cases}$$

and

$$P(t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0, \\ \int_{t_0}^{\vartheta} p(s)du(s), & t_0 \le \vartheta \le t < +\infty, \\ \int_{t_0}^{t} p(s)du(s), & t \le \vartheta < +\infty. \end{cases}$$

Then,

$$G\,(y,t) = F\,(y,t) + P\,(t) \tag{4}$$

defines an element $G\,(y,t)$ from $BG^{-}\left([t_0 - r, +\infty), \mathbb{R}^n\right)$ and $G\,(y,t)\,(\vartheta) \in \mathbb{R}^n$ is the value of $G\,(y,t)$ at the point $\vartheta \in [t_0 - r, +\infty)$, which means,

$$G : O \times [t_0, +\infty) \to BG^{-}([t_0 - r, +\infty), \mathbb{R}^n).$$

Consider the following GODE

$$\frac{dx}{d\tau} = DG(x,t), \tag{5}$$

where the function G is given by (4).

Theorem (Correspondence between the equations)

Let $O \subset G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ with the prolongation property, $S = \{x_t;\ x \in O,\ t \in [t_0, t_0 + \sigma]\}$ and $\phi \in S$. Suppose that $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ and $u : [t_0, t_0 + \sigma] \to \mathbb{R}$ are nondecreasing functions, $f : S \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions (H_1), (H_2), (H_3) and $p : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions (H_4) and (H_5).

(i) Let $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ be a solution of the measure functional differential equation with perturbations

$$\begin{cases} Dy = f(y_t, t)Dg + p(t)Du, & t \in [t_0, t_0 + \sigma], \\ y_{t_0} = \phi. \end{cases} \tag{6}$$

For every $t \in [t_0, t_0 + \sigma]$, let

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

Theorem (Correspondence between the equations)
Then the function $x : [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ is a solution of the GODE (5) with

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), \ t_0 - r \leq \vartheta \leq t_0, \\ \phi(0), \ t_0 \leq \vartheta < t_0 + \sigma. \end{cases}$$

Theorem (Correspondence between the equations)

(ii) Reciprocally, let G be given by (4). Suppose that $x : [t_0, t_0 + \sigma] \to O$ is a solution of the GODE

$$\frac{dx}{d\tau} = DG\,(x, t)\,,$$

with the following initial condition

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), \ t_0 - r \leq \vartheta \leq t_0, \\ \phi(0), \ t_0 \leq \vartheta < t_0 + \sigma. \end{cases}$$

Theorem (Correspondence between the equations)

Then the function $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ defined by

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \leq \vartheta \leq t_0 \\ x(\vartheta)(\vartheta), & t_0 \leq \vartheta < t_0 + \sigma \end{cases}$$

is a solution of the measure functional differential equation with perturbations

$$\begin{cases} Dy = f(y_t, t)Dg + p(t)Du, \quad t \in [t_0, t_0 + \sigma], \\ y_{t_0} = \phi. \end{cases} \tag{7}$$

Lyapunov stability for GODEs

Let X be a Banach space and $B_c = \{x \in X; \|x\| < c\}$, $c > 0$. Define $\Omega = B_c \times [t_0, \infty)$ and let $F : \Omega \to X$.
Consider the GODE

$$\frac{dx}{d\tau} = DF(x(\tau), t) \tag{8}$$

where we suppose that $F(0, t) - F(0, s) = 0$ for $t, s \geq t_0$. Then, $\forall\ [\gamma, v] \subset [t_0, +\infty)$,

$$\int_\gamma^v DF(0, t) = F(0, v) - F(0, \gamma) = 0$$

and, therefore, $x \equiv 0$ is a solution of (8) in $[t_0, +\infty)$.

The trivial solution $x \equiv 0$ of (8) is

(i) Regularly stable, if $\forall\ \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, with $t_0 \leq \gamma < v < +\infty$, is a regulated function which satisfies

$$\|\overline{x}(\gamma)\| < \delta \ \ \text{and} \quad \sup_{s\in[\gamma,v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right\| < \delta,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v].$$

(ii) Regularly attracting, if $\exists\ \delta_0 > 0$ and $\forall\ \varepsilon > 0$, $\exists$ $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, with $t_0 \leq \gamma < v < +\infty$, is a regulated function satisfying

$$\|\overline{x}(\gamma)\| < \delta_0 \quad \text{and} \quad \sup_{s\in[\gamma,v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right\| < \rho,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad \text{for } t \in [\gamma, v] \cap [\gamma + T, +\infty) \text{ and } \gamma \geq t_0.$$

(iii) Regularly asymptotically stable, if it is regularly stable and regularly attracting.

Definition

We say that $V : [t_0, +\infty) \times X \to \mathbb{R}$ is a Lyapunov functional (with respect to the GODE (8)), if the following conditions are satisfied:

(i) $V(\cdot, x) : [t_0, +\infty) \to \mathbb{R}$ is left-continuous in $(t_0, +\infty)$, $\forall$ $x \in X$;

(ii) $\exists$ a function $b : \mathbb{R}^+ \to \mathbb{R}^+$, continuous and strictly increasing, satisfying $b(0) = 0$ (we say that such function is of Hahn class), such that

$$V(t, x) \geq b(\|x\|),$$

$\forall$ $t \in [t_0, +\infty)$ and $x \in X$;

(iii) $\forall\ \bar{x} : [\gamma, v] \to X$ solution of (8), with $[\gamma, v] \subset [t_0, +\infty)$, we have

$$\dot{V}(t, x(t)) = \limsup_{\eta \to 0^+} \frac{V(t+\eta, \bar{x}(t+\eta)) - V(t, \bar{x}(t))}{\eta} \leq 0,$$

t $\in [\gamma, v]$.

Theorem

Let $V : [t_0, +\infty) \times \overline{B_\rho} \to \mathbb{R}$ be a Lyapunov functional, where $\overline{B_\rho} = \{y \in X : \|y\| \leq \rho\}$, $0 < \rho < c$. Suppose that V satisfies the following conditions:

(i) $V(t, 0) = 0,\ t \in [t_0, +\infty)$;

(ii) There exists a constant $K > 0$ such that

$$|V(t,z) - V(t,y)| \leq K\|z - y\|, \quad t \in [t_0, +\infty),\ z, y \in \overline{B_\rho}.$$

Then the trivial solution $x \equiv 0$ of (8) is regularly stable.

Theorem

Let $V : [t_0, +\infty) \times \overline{B_\rho} \to \mathbb{R}$ be a Lyapunov functional, where $\overline{B_\rho} = \{y \in X : \|y\| \le \rho\}$, $0 < \rho < c$. Suppose V satisfies the conditions (i) and (ii) from the previous Theorem. Moreover, suppose there exists a continuous function $\Phi : X \to \mathbb{R}$, satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$, such that for every solution $x : [\gamma, v] \to B_\rho$ of (8), with $[\gamma, v] \subset [t_0, +\infty)$, we have

$$\dot{V}(t, x(t)) \le -\Phi(x(t)), \quad t \in [\gamma, v]. \tag{9}$$

Then the trivial solution $x \equiv 0$ of (8) is regularly asymptotically stable.

Lyapunov stability for measure FDEs

Consider the measure FDE

$$Dy = f(y_t, t)Dg, \tag{10}$$

with $f : S \times [t_0, +\infty) \to \mathbb{R}^n$, where $S = \{x_t;\ x \in O,\ t \in [t_0, +\infty)\}$ e $O \subset BG([t_0 - r, +\infty), \mathbb{R}^n)$ has the prolongation property.

We also consider $g : [t_0, +\infty) \to \mathbb{R}$ nondecreasing and $f(0, t) = 0$ for every $t \in [t_0, +\infty)$ and f satisfies conditions (H_1)-(H_3). Thus $y \equiv 0$ is a solution of (10).

Definition

The trivial solution $y \equiv 0$ of (10) is

(i) *Stable in Lyapunov's sense*, if $\forall\ \varepsilon > 0$ and every $\gamma \in \mathbb{R}$, $\gamma \geq t_0$, $\exists\ \delta = \delta(\varepsilon, \gamma) > 0$ such that, if $\phi \in S$ and $\overline{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [t_0, +\infty)$, is a solution of (10) such that $\overline{y}_\gamma = \phi$ and

$$\|\phi\|_\infty < \delta,$$

then

$$\|\overline{y}_t(\gamma, \phi)\|_\infty < \varepsilon, \quad t \in [\gamma, v].$$

(ii) *Uniformly stable*, if the number δ in the previous item is independent from γ.

Definition

(iii) *Uniformly asymptotically stable*, if $\exists\ \delta_0 > 0$ and $\forall\ \varepsilon > 0$, $\exists$ $T = T(\varepsilon) \geq 0$ such that, if $\phi \in S$, and $\bar{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [t_0, +\infty)$, is a solution of (10) such that $\bar{y}_\gamma = \phi$ and

$$\|\phi\|_\infty < \delta_0,$$

then

$$\|\bar{y}_t(\gamma, \phi)\|_\infty < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty).$$

Definition

We say that $U : [t_0, +\infty) \times G^-([-r,0],\mathbb{R}^n) \to \mathbb{R}$ is a Lyapunov functional (with respect to the measure FDE (10)), if the following conditions hold:

(i) $U(\cdot,\psi) : [t_0,+\infty) \to \mathbb{R}$ is left-continuous in $(t_0,+\infty)$, for every $\psi \in G^-([-r,0],\mathbb{R}^n)$;

(ii) There exists a function of Hahn class $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$U(t,\psi) \geq b(\|\psi\|),$$

for every $t \geq t_0$ and $\psi \in G^-([-r,0],\mathbb{R}^n)$;

(iii) The inequality

$$D^+U(t,\psi) \leq 0$$

holds for each $t \geq t_0$ and $\psi \in G^-([-r,0],\mathbb{R}^n)$.

Let $t \geq t_0$ and $\psi \in G^-([-r,0],\mathbb{R}^n)$. We denote by $y(t,\psi)$ the solution of MFDE (f) with initial condition $y_t = \psi$ and x_ψ the solution of the GODE $\frac{dx}{d\tau} = DG(x,t)$ with initial condition $x_\psi(t) = \widetilde{x}$, where $\widetilde{x}(\tau) = \psi(\tau - t)$, $t - r \leq \tau \leq t$, and $\widetilde{x}(\tau) = \psi(0)$, $\tau \geq t$. Then

- $(t, x_\psi(t)) \mapsto (t, y_t(t,\psi))$ is a one-to-one application.

- We define $V : [t_0, +\infty) \times O \to \mathbb{R}$ by

$$V(t, x_\psi(t)) = U(t, y_t(t,\psi)).$$

Then the righthand derivative $D^+U(t,\psi)$ can be written

$$D^+U(t,\psi) = \lim_{\eta \to 0^+} \sup \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta}, \qquad t \geq t_0.$$

We obtain the following theorems for MFDEs:

Theorem
Consider the measure functional differential equation (10). Suppose the function $f : S \times [t_0, \infty) \to \mathbb{R}^n$ satisfies the conditions (H_1), (H_2) and (H_3) and $U : [t_0, +\infty) \times \overline{E}_\rho \to \mathbb{R}$ is a Lyapunov functional. Moreover, assume that the following conditions are satisfied:

(i) $U(t, 0) = 0, \ t \in [t_0, +\infty)$;

(ii) There exists a constant $K > 0$ such that

$$|U(t,\psi) - U(t,\overline{\psi})| \leq K\|\psi - \overline{\psi}\|, \quad t \in [t_0, +\infty), \ \psi, \overline{\psi} \in \overline{E}_\rho.$$

Then the trivial solution $y \equiv 0$ of (10) is uniformly stable.

Idea of the proof

Define

- The solution $y \equiv 0$ of (10) is said to be *integrally stable*, if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $\psi \in S$ with $\|\psi\|_\infty < \delta$ and

  $$\sup_{t \in [\gamma, v]} \left| \int_\gamma^t p(s) \mathrm{d}u(s) \right| < \delta,$$

  where $t_0 \leq \gamma \leq v < \infty$, then

  $$\|\bar{y}_t(\gamma, \psi)\|_\infty < \varepsilon, \quad \text{for every} \quad t \in [\gamma, v],$$

  where $\bar{y}(t; \gamma, \psi)$ is a solution of the MFDE with perturbations satisfying $\bar{y}_\gamma = \psi$.

Idea of the proof

- The solution $y \equiv 0$ of (10) is called *integrally attracting*, if there is a $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if

$$\|\psi\|_\infty < \widetilde{\delta} \qquad \text{and} \qquad \sup_{t\in[\gamma,v]} \left| \int_\gamma^t p(s)\mathrm{d}u(s) \right| < \rho,$$

  where $t_0 \leq \gamma \leq v < \infty$, then

$$\|\bar{y}_t(\gamma,\psi)\|_\infty < \varepsilon \ \text{ for all } \ t \geq \gamma + T, \ t \in [\gamma, v],$$

  where $\bar{y}(t;\gamma,\psi)$ is a solution of the MFDE with perturbations satisfying $\bar{y}_\gamma = \psi$.
- The solution $y \equiv 0$ of (10) is called *integrally asymptotically stable*, if it is integrally stable and integrally attracting.

Idea of the proof

Since we have $U(t, y_t(t, \psi)) = V(t, x_\psi(t))$, one can show that the trivial solution of the GODE related to the MFDE is regularly stable.

Theorem

Regular stability $\Rightarrow$ Integral stability

Idea of the proof

Since we have $U(t, y_t(t, \psi)) = V(t, x_\psi(t))$, one can show that the trivial solution of the GODE related to the MFDE is regularly stable.

Theorem

Regular stability $\Rightarrow$ Integral stability

Theorem

Consider the measure functional differential equation (10). Suppose $U : [t_0, +\infty) \times \overline{E_\rho} \to \mathbb{R}$ is a Lyapunov functional and satisfies conditions (i) and (ii) from the previous Theorem. Furthermore, suppose there exists a continuous function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Lambda(0) = 0$ and $\Lambda(x) > 0$ if $x \neq 0$, such that, for every $\psi \in \overline{E}_\rho$, we have

$$D^+ U(t, \psi) \leq -\Lambda(\|\psi\|), \quad t \geq t_0. \tag{11}$$

Then, the trivial solution $y \equiv 0$ of (10) is uniformly asymptoticaly stable.

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