C-Removable sets

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(joint work with Dušan Pokorný)

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Definition (J. Tabor & J.Tabor)

We call $A \subset \mathbb{R}^n$ intervally thin if for all $x, y \in \mathbb{R}^n$ and $\varepsilon > 0$ there exist $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$ such that $[x', y'] \cap A = \emptyset$.

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For $v \in \mathbb{R}^n \setminus \{0\}$ we say that A is intervally thin in the direction v if the above condition is satisfied for any $x, y \in \mathbb{R}^n$ such that $x - y = cv$ for some $c \in \mathbb{R}$.

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Let $A \subset \mathbb{R}^n$ be closed. Suppose that, for an arbitrary open set U containing A, every preconvex function $f : U \setminus A \to \mathbb{R}$ has a unique extension on U. Is it then necessarily true that A is intervally thin?

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Of course, the answer is NO.

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Also note that a preconvex function on an open set (i.e. convex on any line segment) is locally Lipschitz.

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Which sets are C-removable?

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(v) The ugly counterexample. $\Longrightarrow$ Draw an ugly picture.

Pasqualini, 1938

Let $M \subset \mathbb{R}^n$ be a set which does not contain any continuum of topological dimension $(n-1)$. If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function which is preconvex on $\mathbb{R}^n \setminus M$ then f is convex on R^n.

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Note that the last statement also follows from the fact that Pasqualini's proof contains a mistake which cannot be repaired. In addition, it is a consequence of the following theorem.

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There exists a zero-dimensional compact set $K \subset \mathbb{R}$ such that K^2 is not C-removable. (Here $\lambda(K) > 0$.)

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Open problem

Is there a zero-dimensional compact in $\mathbb{R}^2$ of Lebesgue measure zero, which is not C-removable? If so, then what is the minimal (infimal) possible Hausdorff dimension of such a set?

It can be anywhere between 1 and 2.

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Question: Is there a non-trivial C-removable continuum? The Koch snowflake?

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- WLOG f is separately convex and is not convex on the line L given by $y = x$. Assume also that L happens to have countable intersection with K.
- Lemma: A separately convex function cannot have a concave angle.
- $L \cap K$ is a countable compact set, so it is scattered. Deduce that f is convex on L.

Thank you for your attention.