COMPARISON OF SOME TRIGONOMETRIC INTEGRALS

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THEOREM (VALLÉE-POUSSIN). *If f is a finite, Lebesgue integrable on $[0, 2\pi]$ function and there is a trigonometric series*

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{1}$$

convergent to f nearly everywhere on $[0, 2\pi]$, then it is the Fourier series of f.

EXAMPLE (DENJOY):

$$\sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n \searrow 0, \quad \sum_{n=1}^{\infty} \frac{b_n}{n} = \infty,$$

is a series convergent everywhere to a function that **is not Lebesgue integrable**.

Does there exist an integral by means of which the coefficients of each everywhere convergent trigonometric series can be recovered by Fourier formulas?

RIEMANN THEORY

$$F(x) = c_0 \frac{x^2}{2} - \sum_{n \neq 0} \frac{c_n}{n^2} e^{inx} \qquad \text{is continuous on} \quad [0, 2\pi].$$

THEOREM (RIEMANN). *If* (1) *sums at* x *to a finite number* s, $c_n \to 0$, *then there exists finite*

$$D^2 F(x) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2},$$

called *the second symmetric derivative of* F, and it equals s.

THEOREM (RIEMANN). *If* $c_n \to 0$ *in* (1), *then*

$$\frac{F(x+h) - 2F(x) + F(x-h)}{h} \rightrightarrows 0, \qquad h \to 0, \quad x \in [0, 2\pi].$$

$$l(x) = c_0 x + \sum_{n\neq 0} \frac{c_n}{in} e^{inx} \qquad \text{is finite almost everywhere on} \quad [0, 2\pi].$$

DEFINITION. Let F be defined on a measurable set E. We say F is *approximately continuous at x if*

$$\operatorname{ap-lim}_{h\to x} F(h) = F(x),$$

i.e. $F(h)$ tends to $F(x)$ as $h \in E'$ tends to x, $E' \subset E$ with density 1 at x.

DEFINITION. A function F defined on a measurable set E is called *approximately symmetrically continuous at x*, if

$$\operatorname{ap-lim}_{h\searrow 0}(F(x+h) - F(x-h)) = 0.$$

LEBESGUE THEORY

DEFINITION. A function F defined on a measurable set E is called *approximately symmetrically differentiable at x*, if there exists a finite limit

$$\operatorname{ap-lim}_{h \searrow 0} \frac{F(x+h) - F(x-h)}{2h} = F'_{\text{sap}}(x).$$

THEOREM (ZYGMUND). *If (1) sums at x to a finite number s, $c_n \to 0$, then there exists $l'_{\text{sap}}(x)$ and it equals s.*

THEOREM (ZYGMUND–RAJCHMAN). *If $c_n \to 0$ in (1), then l is approximately symmetrically continuous and approximately continuous at every point where l is finite.*

Integrals which solve the problem of recovery on the base of the Riemann theory:

Denjoy: totalization T_{2s} integration process

Burkill: SCP-integral

Marcinkiewicz–Zygmund: $T(P)$-integral

James: P^2-integral

An integral which solves the problem of recovery on the base of the Lebesgue theory:

Preiss–Thomson: AS-integral

BURKILL'S SCP-INTEGRAL

DEFINITION. Let G be Perron integrable function with an indefinite integral F. Then G is called SC-lower (upper) semicontinuous at x if

$$\liminf_{h\to 0+} \frac{1}{h}\left(\int_x^{x+h} - \int_{x-h}^{x}\right) G(t)\,dt = \liminf_{h\to 0+} \frac{F(x+h) - 2F(x) + F(x-h)}{h} \geqslant 0 \quad (\leqslant 0).$$

G is called C-continuous at x if $G(x) = F'(x)$ (at end points – in one-sided sense).

Symmetric lower and upper Cesàro derivatives of G at x are

$$\underline{SCD}\, G(x) = \underline{D}^2 F(x), \qquad \overline{SCD}\, G(x) = \overline{D}^2 F(x).$$

SCD-derivative of G at x is

$$SCD\, G(x) = \lim_{h\to 0+} \frac{1}{h^2}\left(\int_x^{x+h} - \int_{x-h}^{x}\right) G(t)\,dt = D^2 F(x).$$

BURKILL'S SCP-INTEGRAL

DEFINITION. Let f be a finite function on (a, b) and B a set of full measure on $[a, b]$ containing a, b. A function M is said to be an SCP-major function of f on $[a, b]$ with basis B if

1. $M(a) = 0$.
2. M is SC-lower semicontinuous everywhere on (a, b).
3. M is C-continuous in B.
4. $\underline{SCD}\, M(x) \geqslant f(x)$ for almost every $x \in (a, b)$.
5. $\underline{SCD}\, M(x) > -\infty$ for nearly every $x \in (a, b)$.

A function m is said to be an SCP-minor function for f on $[a, b]$ with basis B if $-m$ is an SCP-major function for $-f$ on $[a, b]$ with basis B.

MONOTONICITY THEOREM. *Let F be a continuous function on (a, b) and $\underline{D}_2 F(x) \geqslant 0$ a.e. and $\underline{D}_2 F(x) > -\infty$ nearly everywhere and*

$$\liminf_{h \to 0+} \frac{F(x+h) - 2F(x) + F(x-h)}{h} \geqslant 0 \qquad \text{for all} \quad x,$$

then F is convex.

$$\int (M - m) \text{ is convex} \qquad \Rightarrow \qquad M - m \text{ is nondecreasing a.e.}$$

BURKILL'S *SCP*-INTEGRAL

If

$$I = \inf M(b) = \sup m(b),$$

where inf is taken over all SCP-major functions M for f and sup over all SCP-minor functions m for f then f is called SCP-integrable on $[a, b]$ with basis B and we write

$$I = (SCP, B) \int_a^b f(x)\, dx.$$

$(SCP, B) \int_a^x f(t)\, dt$ is called indefinite SCP-integral with basis B of f.

THEOREM (BURKILL). *If* (1) *converges nearly everywhere to a finite function* f*, then* f *is* SCP*-integrable and*

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-inx} dx,$$

for any basis $B \ni 0, 2\pi$*.*

PREISS–THOMSON'S *AS*-INTEGRAL

Let $\mathcal{I}$ be the set of all nondegenerate closed intervals on $\mathbb{R}$.

DEFINITION. A set $\beta \subset \mathcal{I} \times \mathbb{R}$ is called *measurable approximate symmetric element*, if

(1) for each pair $(I, x) \in \beta$ the interval I is symmetric with respect to x, i.e. $I = [x-t, x+t]$, and $T = \{(x,t) : ([x-t, x+t], x) \in \beta\}$ is a measurable set on $\mathbb{R} \times (0, \infty)$,

(2) for all $x \in \mathbb{R}$: $\displaystyle \lim_{h \searrow 0} \mu\big(\{t \in (0,h) \colon (x,t) \notin T\}\big)/h = 0.$

A finite collection of pairs $\pi = \{(I, x)\}$ is called a *division* (*a partition*) of interval $[a, b]$ if for any different pairs (I_1, x_1), (I_2, x_2) from π intervals I_1 and I_2 do not overlap (and $\bigcup I = [a, b]$).

PARTITIONING THEOREM. *For any measurable approximate symmetric element β almost every closed interval can be partitioned with tagged intervals from β.*

PREISS–THOMSON'S *AS*-INTEGRAL

Let us denote by $\mathcal{A}_B$ a set of all measurable approximate symmetric elements containing at least one partition of every closed interval with endpoints in B.

DEFINITION. A function f, defined everywhere on $\mathbb{R}$, is called *AS-integrable*, if there exists a set B of the full measure and a function F on B such that for any $\varepsilon > 0$ there is an element $\beta \in \mathcal{A}_B$ such that for any division $\pi = \{([y_i, z_i], (y_i + z_i)/2)\} \subset \beta$ on the line, with $y_i, z_i \in B$, the inequality

$$\left|\sum_{\pi}\big(f((y_i+z_i)/2)(z_i-y_i)-(F(z_i)-F(y_i))\big)\right| < \varepsilon$$

holds. For each pair $a, b \in B$ the number $F(b) - F(a)$ will be denoted $(AS)\int\limits_a^b f(x)dx$.

The function $F(x)$, defined on B, is called *indefinite AS-integral* of the function f.

PREISS–THOMSON'S *AS*-INTEGRAL

THEOREM (PREISS–THOMSON). *If a function F is measurable, approximately symmetrically continuous at each point of the line and has nearly everywhere approximate symmetric derivative f, then the function $f = F'_{\mathrm{sap}}$ is AS-integrable with F being an indefinite integral.*

THEOREM (PREISS–THOMSON). *If* (1) *converges nearly everywhere to a finite function f, then functions $f(x)$ and $f(x)e^{-inx}$, $n \in \mathbb{Z}$, are AS-integrable and*

$$c_n = \frac{1}{2\pi} \int\limits_{p}^{p+2\pi} f(x)\, e^{-inx} dx$$

for almost all p.

COMPARISON OF SCP- AND AS-INTEGRALS

$AS \not\Rightarrow SCP$

Take

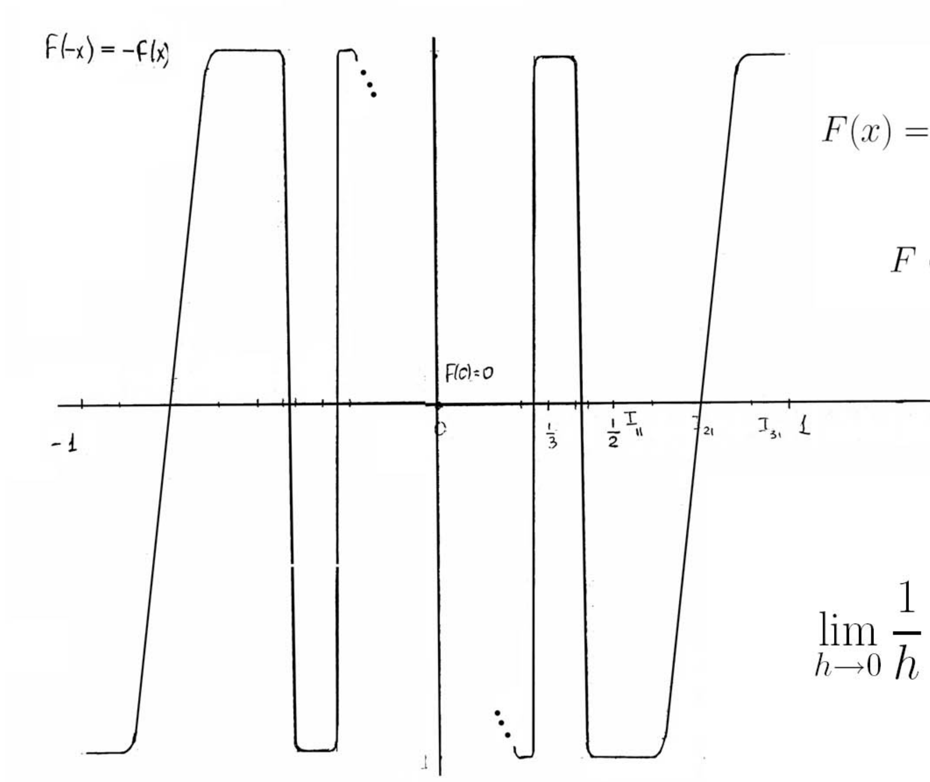
$$F(x) = \frac{1}{x^2}\,, \qquad f(x) = \begin{cases} F'(x) = -2/x^3, & x \neq 0, \\ F'_{\text{sap}}(x) = 0, & x = 0. \end{cases}$$

f is ***AS*-integrable**.

F is not Perron integrable on $[-1,1]$. $F-1$ satisfies conditions 1.–5. for SCP-major functions for f. So any other function with 1.–5. would be $\geqslant F-1$ and it will not be Perron integrable as well.

f is not ***SCP*-integrable**.

$SCP \not\Rightarrow AS$



$$F(x)=\begin{cases}(-1)^n, & x\in I_{1n},\\ F(x), & x\in I_{2n},\\ (-1)^{n+1}, & x\in I_{3n},\end{cases}\qquad |F(x)|\geqslant 1,\quad x\in I_{2n}.$$

$$F\in C^1(1/(n+1),1/n),\qquad \int_{1/(n+1)}^{1/n}F(t)\,dt=0.$$

$$\lim_{h\to 0}\frac{1}{h}\int_0^h F(t)\,dt=\lim_{h\to 0}\frac{1}{h}\int_{-h}^0 F(t)\,dt=0.$$

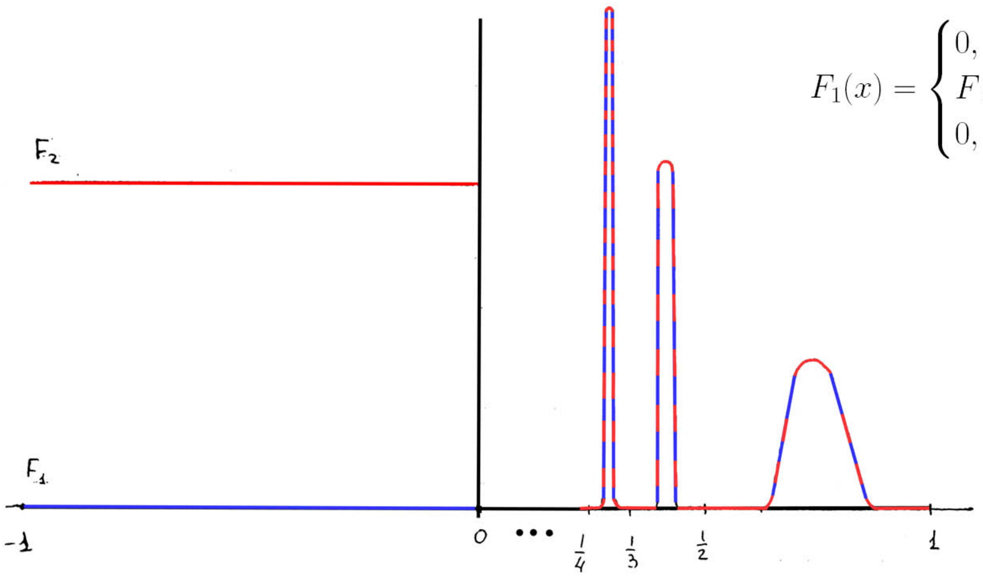
$|I_{2n}|=(2^n n(n+1))^{-1}$.

F is the ***SCP*-integral** for F', which exists everywhere except 0.

The set $\{h>0:|F(h)-F(-h)|\geqslant 1\}$ has density 1 at 0 so F is not AS-continuous at 0, so F' is not ***AS*-integrable**.

COMPARISON OF *SCP*- AND *AS*-INTEGRALS

Compatibility:



$$F_1(x) = \begin{cases} 0, & x \in I_{1n}, \\ F(x), & x \in I_{2n}, \\ 0, & x \in I_{3n}, \end{cases} \qquad \int_{1/(n+1)}^{1/n} F(x)\,dt = 1/n - 1/(n+1).$$

$$F \in C^1(1/(n+1), 1/n)$$

$$F_1(x) = 0,\ x \in [-1, 0]$$

$$F_2(x) = \begin{cases} F_1(x), & x \in (0, 1], \\ 1, & x \in [-1, 0]. \end{cases}$$

F_1 is AS-continuous at 0. $\qquad$ F_2 is SC-continuous at 0.

$$f = F_1' = F_2'.$$

F_1 is the AS-integral for f and F_2 is the SCP-integral for f over $[-1, 1]$.

COMPARISON OF SCP- AND AS-INTEGRALS

THEOREM. *There exists an AS- and SCP-integrable function with* **continuous** *AS-indefinite integral F_1 and* **continuous** *SCP-integral F_2 such that $F_1 - F_2$ is a* **continuous Cantor-type singular function.**