Perturbation of Dynamical Systems defined by Semilinear Parabolic Problems

MARCONE C. PEREIRA[1]

Universidade de São Paulo
São Paulo – Brazil

SYMPOSIUM IN REAL ANALYSIS XXXVII,
SÃO CARLOS, SP, BRAZIL

[1] Partially supported by FAPESP 2008/53094-4 and 2010/18790-0, CAPES DGU 127/07 and CNPq 302847/2011-1.

1. Initially we establish some abstract results for a class of semilinear problems defined in Banach spacs.
2. Next we apply them to **gradient systems** given by partial differential equations with **nonlinear boundary conditions** in the following situations:
   - **(i)** when the domain of definition of the solutions vary with respect to the action of diffeomorphisms;
   - **(ii)** when some reaction and potential terms of the equation are concentrating in a narrow strip of a portion of the boundary of the domain of the solutions.

Our **main goal** is to discuss the continuity of the nonlinear semigroup, as well as, the upper and lower semicontinuity of the family of attractors.

★ We recall that an *attractor is a compact invariant set which attracts the flow for all bounded sets of the phase space*. It contains all the asymptotic dynamics of the system and all global bounded solutions lie in the attractor.

The study of the lower semicontinuity of attractors for semilinear differential equations in Banach spaces has its origin in the work of

[J. K. Hale and G. Raugel, Lower semicontinuity of attractors of gradient systems and applications, Ann. Mat. Pura Appl. (1989)]

where an abstract result has been proved and applications to partial differential equations have been considered. The results in that paper that have been used and simplified since then says that:

If the limiting equation is gradient, has a finite number of equilibria, all of them hyperbolic, the perturbed nonlinear semigroups vary continuously, the sets of equilibria have fixed finite cardinality and vary continuously with the parameter, and the local unstable manifolds of the perturbed problems are lower semicontinuous, then the family of attractors behaves lower semicontinuously.

There are several works in the literature about continuity of the dynamics generated by parabolic problems autonomous and non-autonomous. For instance let us mention **[Arrieta, Carvalho, Langa, Rodríguez-Bernal, J. of Dyn. Syst. and Partial Diff. Eq. (2012)]** as a nice review and new applications.

Some abstract results.

Let us consider the following semilinear equation

$$(P_\lambda) \qquad \frac{dx}{dt} = A_\lambda x + f(t, x, \lambda), \quad t > t_0, \qquad x(t_0) = x_0.$$

We assume:

- $\{-A_\lambda\}_{\lambda \in \Lambda}$ is a family of operators in a Banach space X, with $A_{\lambda_0} = A$ for $\lambda_0 \in \Lambda$, where Λ is an open set of a metric space Y.
- A is a sectorial operator with $\|(\mu - A)^{-1}\| \leq \frac{M}{|\mu - a|}$ for all μ in the sector $S_{a,\phi_0} = \{\mu \mid \phi_0 \leq |arg(\mu - a)| \leq \pi, \mu \neq a\}$, for $a \in \mathbb{R}$ and $0 \leq \phi_0 < \pi/2$, such that, the fractional power spaces X^α are well defined.

Also we suppose the family $\{-A_\lambda\}_{\lambda \in \Lambda}$ satisfies

1. $D(A_\lambda) \supset D(A)$, for all $\lambda \in \Lambda$;
2. $\|A_\lambda x - Ax\| \leq \epsilon(\lambda)\|Ax\| + K(\lambda)\|x\|$ for any $x \in D(A)$, where $K(\lambda)$ and $\epsilon(\lambda)$ are positive functions with $\lim_{\lambda \to \lambda_0} \epsilon(\lambda) = 0$ and $\lim_{\lambda \to \lambda_0} K(\lambda) = 0$.

★ This kind of linear perturbation was deeply studied in **[T. Kato, Perturbation theory for linear operators, Springer-Verlag, (1980)]**.

- We suppose
$$f : U \times \Lambda \mapsto X$$
is Hölder continuous in t, where U is an open set in $\mathbb{R}^+ \times X^\alpha$, $0 \leq \alpha < 1$.
- Also, for any bounded subset $D \subset U$, we assume f is continuous in $\lambda = \lambda_0$, uniformly for (t, x) in D, and there is a constant $L = L(D)$, such that
$$\|f(t, x, \lambda) - f(t, y, \lambda)\| \leq L\|x - y\|_\alpha$$
for (t, x), (t, y) in D and $\lambda \in \Lambda$.

Let us just observe that the framework on semilinear parabolic equations that we are setting here fits into **[D. Henry, Lecture Notes in Math. 840, (1981)]**.

In the end, we are taking a class of perturbations of the semilinear problem

$$(P_0) \qquad \begin{aligned} \frac{dx}{dt} &= Ax + f(t, x, \lambda_0), \quad t > t_0, \\ x(t_0) &= x_0. \end{aligned}$$

Under these conditions, we can work on results from [T. Kato, (1980)] and [D. Henry, (1981)] to show the existence of a neighborhood V of λ_0 such that

- A_λ **is sectorial** for $\lambda \in V$. In fact, we get

$$\|(\mu - A_\lambda)^{-1}\| \leq C\,\|(\mu - A)^{-1}\|$$

  for some constant $C > 0$, for all $\lambda \in V$ and $\mu \in S_{b,\phi}$ where

$$S_{b,\phi} = \{\mu \in \mathbb{C} \mid \phi \leq |arg(\mu - b)| \leq \pi, \mu \neq b\}$$

  with $b \simeq a$, and ϕ independent of λ.
- We also get the **convergence of the resolvent operators** by

$$\|(\mu - A_\lambda)^{-1} - (\mu - A)^{-1}\| \quad \leq \|(\mu - A_\lambda)^{-1}\|\|(A - A_\lambda)\cdot(\mu - A)^{-1})\|$$

  for each $\mu \in S_{b,\phi}$.
- Then we show that the family of **linear semigroups** e^{-tA_λ} satisfy

$$\|e^{-tA_\lambda} - e^{-tA}\| \leq C(\lambda)e^{-bt}$$

$$\|A\left(e^{-tA_\lambda} - e^{-tA}\right)\| \leq C(\lambda)\frac{1}{t}e^{-bt}$$

$$\|A^\alpha\left(e^{-tA_\lambda} - e^{-tA}\right)\| \leq C(\lambda)\frac{1}{t^\alpha}e^{-bt}, \quad 0 < \alpha < 1$$

  for $t > 0$ with $C(\lambda) \to 0$ as $\lambda \to \lambda_0$.

- If $\sigma(A) = \sigma_1 \cup \sigma_2$, where σ_1 and $\sigma_2 \cup \{\infty\}$ are spectral sets, σ_1 bounded, with $\mathrm{Re}(\sigma_1) < -\alpha < 0$ and $\mathrm{Re}(\sigma_2) > \beta > 0$. Then, if $|\lambda - \lambda_0|$ is sufficiently small, we also have $\sigma(A_\lambda) = \sigma_{1\lambda} \cup \sigma_{2\lambda}$, Re $(\sigma_{1\lambda}) < -\alpha < 0$ and Re $(\sigma_{2\lambda}) > \beta > 0$. If we denote by E_1 (resp. $E_{1\lambda}$), E_2 (resp. $E_{2\lambda}$) the corresponding spectral projections, then

  $$||E_{j\lambda} - E_j|| \to 0 \quad \text{and} \quad ||A(E_{j\lambda} - E_j)|| \to 0 \quad \text{as} \quad \lambda \to \lambda_0, \quad \text{for} \quad j = 1, 2.$$

- Suppose the nonlinear semigroup $T(t, \lambda)$ of the problem

  $$\frac{dx}{dt} = A_\lambda x + f(t, x, \lambda), \quad t > t_0$$
  $$x(t_0) = x_0$$

  is defined for each x_0 in bounded subsets of X^α, λ in a neighborhood of λ_0 and $t_0 \le t \le t_1$.

Then the function $\lambda \mapsto T(t, \lambda)x_0 \in X^\alpha$ is continuous at λ_0 uniformly for x_0 in bounded subsets of X^α and $t_0 \le t \le t_1$.

At the end, the convergence of the linear semigroup is transferred to the nonlinear dynamics through the variation of constant formula.

We recall that the family of subsets $\mathcal{A}_\lambda$ of a metric space (X, d) is said to be *upper-semicontinuous* at $\lambda = \lambda_0$ if

$$\delta(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \to 0 \text{ as } \lambda \to \lambda_0,$$

where

$$\delta(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$$

and *lower-semicontinuous* if

$$\delta(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) \to 0 \text{ as } \lambda \to \lambda_0.$$

- Suppose also that, for each $\lambda \in \Lambda$ there exists a global compact attractor $\mathcal{A}_\lambda$ for $T(t, \lambda)$, the union $\cup_{\lambda \in \Lambda} \mathcal{A}_\lambda$ is a bounded set in X and f maps this union into a bounded set of X.

Then the family $\mathcal{A}_\lambda$ is upper semicontinuous.

- Suppose in addition that the **system generated is gradient** for any λ and their **equilibria are all hyperbolic and continuous in** λ.

Then the family $\mathcal{A}_\lambda$ is also lower semicontinuous.

Indeed, the global attractor $\mathcal{A}_\lambda \subset \mathcal{H}$ captures all the asymptotic dynamics of the system, and satisfies $\mathcal{A}_\lambda = W^u(\mathcal{E}_\lambda)$, where $\mathcal{E}_\lambda$ is the set of equilibria of (P_λ) given by

$$\mathcal{E}_\lambda = \{\varphi \in \mathcal{H} \, : \, T(t,\lambda)\varphi = \varphi \text{ for all } t \geq 0\},$$

and $W^u(\mathcal{E}_\lambda)$ denotes the unstable set of $\mathcal{E}_\lambda$. Moreover, if each equilibrium is hyperbolic, then $\mathcal{E}_\lambda$ is finite and

$$\mathscr{A}_\lambda = \cup_{\varphi \in \mathcal{E}_\lambda} W^u(\varphi),$$

where $W^u(\varphi)$ denotes the unstable manifold of the equilibrium point φ.

At the end, the continuity of the unstable manifold ensures the lower semicontinuity of the attractors.

For proofs see **[A. Pereira, M. Pereira, JDE (2007)]**.

First example: We consider the dependence of the dynamics defined by

$$\begin{cases} v_t(y,t) = \Delta v(y,t) - a\,v(y,t) + f(v(y,t)), & y \in \Omega_h, \\ \frac{\partial v}{\partial N}(y,t) = g(v(y,t)), & y \in \partial\Omega_h, \end{cases} \quad t > 0, \qquad (1)$$

with respect to perturbations of a C^m-bounded domain $\Omega \subset \mathbb{R}^n$, $m \geq 2$, where

$$\Omega_h = h(\Omega) \quad \text{for} \quad h \in \mathrm{Diff}^m(\Omega),$$

$$\mathrm{Diff}^m(\Omega) = \{h \in C^m(\bar{\Omega}, \mathbb{R}^n) \, : \, h \text{ is injective and } 1/|\det h'| \text{ is bounded in } \bar{\Omega}\}.$$

Under the following growth and dissipative conditions, it has been proved that (1) define a nonlinear gradient semigroup and admit a global attractor.[a]

[a]See [Carvalho, Oliva, A. Pereira, Rodríguez-Bernal, JMAA (1997)], [Oliva, A. Pereira, Dyn. Contin. Impuls. Syst. (2002)] and [Arrieta, Carvalho, Rodríguez-Bernal, Comm. PDE (2000)].

(H1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $g \in C^2(\mathbb{R}, \mathbb{R})$ are bounded functions with bounded derivative such that, there are constants c_0 and d_0 with

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} \leq c_0, \quad \limsup_{|s|\to\infty} \frac{g(s)}{s} \leq d_0.$$

(H2) c_0 and d_0 are such that the first eigenvalue μ_1 of the problem

$$\begin{array}{ll} -\Delta\varphi + (a - c_0)\varphi = \mu\varphi & \text{in } \Omega_h \\ \frac{\partial\varphi}{\partial N_\Omega} = d_0\varphi & \text{on } \partial\Omega_h \end{array}$$

is positive.

There are several works in literature investigating boundary perturbation problems, let us mention just some of them.

- Results on this direction have been obtained in **[D. Daners, JDE (2003)]**, **[de Abreu, Carvalho, Mat. Contemp. (2004)]** and **[Oliveira, Pereira, Pereira, EJDE (2005)]** for the problem with Dirichlet boundary conditions.
- In **[Arrieta, Carvalho, JDE (2004)]** the authors prove continuity of the attractor for Neumann homogeneous boundary conditions and some kinds of singular perturbations of the boundary.
- The continuity of the equilibria of (1) has been considered in **[Arrieta, Bruschi, C. R. Sci. Paris (2006)]**, **[Arrieta, Bruschi, Math. Models Methods Appl. Sci. (2007)]**, **[D. Daners, Handbook Diff. Eq. (2008)]**, and **[Arrieta, Bruschi, Discrete Contin. Dyn. Syst. (2010)]** where the authors also allow some kinds of singular perturbations.
- Recently **[Ngiamsunthorn, Nonlinear Analysis: Theory, Methods & Applications (2013)]** consider the effect of domain perturbation on invariant manifolds for semilinear parabolic equations also in Dirichlet boundary condition.

One of the difficulties here is that the functional spaces change as we change the region. Our first task is then to find a way to compare the attractors of problem (1) in different regions. One possible approach is the one taken by Henry in [D. Henry, Cambridge Univ. Press (2005)].

We express problems of perturbation of the boundary of boundary value problem as problems of differential calculus in Banach spaces.

If $h : \Omega \mapsto \mathbb{R}^n$ is a C^k embedding, we may consider the 'pull-back' of h

$$h^* : C^k(h(\Omega)) \mapsto C^k(\Omega) \quad (0 \leq k \leq m)$$

defined by $h^*(\varphi) = \varphi \circ h$, which is an isomorphism with inverse h^{-1^*}.

Other function spaces can be used instead of C^k, and we will actually be interested mainly in Sobolev spaces and fractional power spaces.

If $F_{\Omega_h} : C^m(\Omega_h) \mapsto C^0(\Omega_h)$ is a nonlinear differential operator in $\Omega_h = h(\Omega)$ we also can consider the differential operator $h^* F_{\Omega_h} h^{*-1}$ for a fixed region Ω.

Consequently, $v(\cdot,t)$ satisfies (1) in Ω_h if and only if $u(\cdot,t)=h^*v(\cdot,t)$ satisfies

$$\begin{cases} u_t(x,t) = h^*\Delta_{\Omega_h} h^{*-1}u(x,t) - a\,u(x,t) + f(u(x,t)), x\in\Omega, \\ h^*\dfrac{\partial}{\partial N_{\Omega_h}} h^{*-1}u(x,t) = g(u(x,t)), x\in\partial\Omega, \end{cases} \quad t>0, \qquad (2)$$

where

$$h^*\Delta_{\Omega_h} h^{*-1}u(x) = \Delta_{\Omega_h}(u\circ h^{-1})(h(x)),$$
$$h^*\frac{\partial}{\partial N_{\Omega_h}} h^{*-1} = \frac{\partial}{\partial N_{\Omega_h}}(u\circ h^{-1})(h(x)).$$

★ In particular, if $\mathcal{A}_h$ is the global attractor of (1) in $H^1(\Omega_h)$, then

$$\hat{\mathcal{A}}_h = \{v\circ h \mid v\in\mathcal{A}_h\}$$

is the global attractor of (2) in $H^1(\Omega)$ and conversely.

In this way, we can consider the problem of continuity of the attractors as $h\to i_\Omega$ in a fixed phase space using standard perturbation theory.

Abstract setting. Writing problem (2) in an abstract form.

Consider now, the operator $A_h = h^* \,(\Delta_h - a)\, h^{*\,-1}$ defined in the fixed region Ω. Since h^* and $h^{*\,-1}$ are isomorphisms (in the appropriate spaces), the operator $-A_h$ is a self-adjoint positive operator in $L^2(\Omega)$, with

$$D(-A_h) = \{u \in H^2(\Omega) \mid h^* \frac{\partial}{\partial N_{h(\Omega)}} h^{*\,-1} u = 0\},$$

where Δ_h is the Laplacian operator with Neumann homogeneous boundary conditions in $\Omega_h = h(\Omega)$.

Hence, integrating by parts, we can define the linear operators $A_h : H^1(\Omega) \subset H^{-1}(\Omega) \mapsto H^{-1}(\Omega)$ by

$$\begin{aligned}\langle A_h u + au, \phi\rangle_{-1,1} &= \int_\Omega \left(h^* \Delta_{h(\Omega)} h^{*\,-1} u\right)(x) \cdot \phi(x)\, dx \\ &= -\int_\Omega h^* \nabla_{h(\Omega)} h^{*\,-1} u(x) \cdot h^* \nabla_{h(\Omega)} h^{*\,-1} \frac{\phi}{|Jh|}(x)\, |Jh(x)|\, dx,\end{aligned}$$

for u, $\phi \in H^1(\Omega)$, where Jh is the determinant of Jacobian matrix h_x.

Now, for each $h \in \mathrm{Diff}^2(\Omega)$ consider the nonlinear operators:

(i) $F_h = F(\cdot, h) : H^r(\Omega) \mapsto H^{-1}(\Omega)$ defined by

$$\langle F_h(u), \phi \rangle_{-1,1} = \int_\Omega f(u)\, \phi \, dx$$

(ii) $G_h = G(\cdot, h) : H^r(\Omega) \mapsto H^{-1}(\Omega)$, $\frac{1}{2} \leq r \leq 1$, defined by

$$\langle G_h(u), \phi \rangle_{-1,1} = \int_{\partial\Omega} \Gamma(g(u))\, \Gamma(\phi) \left| \frac{J_{\partial\Omega} h}{Jh} \right| d\sigma(x)$$

where $\Gamma : H^r(\Omega) \mapsto H^{r-\frac{1}{2}}(\partial\Omega)$ is the trace map.

Thus, if $H_h = F_h + G_h$, we can consider our problem in the abstract form:

$$\dot{u} = A_h u + H_h(u), \quad t > t_0$$
$$u(t_0) = u_0 \in H^r(\Omega), \quad \frac{1}{2} \leq r < 1.$$

Estimating the family of operators $\{-A_h\}_{h\in \mathrm{Diff}^2(\cdot)}$.

It is clear that $-A_{i_\Omega}$ is a sectorial operator and $D(-A_h)\subset D(-A_{i_\Omega})$ for all $h\in \mathrm{Diff}^2(\Omega)$. Then we need to show there exist $\epsilon(h)$, $K(h)>0$ such that

$$\|(A_h - A_{i_\Omega})u\|_{H^{-1}(\Omega)} \leq \epsilon(h)\|A_{i_\Omega}u\|_{H^{-1}(\Omega)} + K(\epsilon)\|u\|_{H^{-1}(\Omega)}, \quad u\in D(-A_{i_\Omega}),$$

where $\lim_{h\to i_\Omega} K(h) = \lim_{h\to i_\Omega}\epsilon(h) = 0$. More precisely, we show

$$|\langle (A_h - A_{i_\Omega})u,\psi\rangle_{-1,1}| \leq \epsilon(h)\,\|u\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)}$$

for all $u\in H^1(\Omega)$ and for all $\psi\in H^1(\Omega)$ with $\lim_{h\to i_\Omega}\epsilon(h)=0$.

Consequently:

- We obtain the convergence of the linear semigroup e^{-tA_h} as $h \to i_\Omega$ in $C^2(\bar{\Omega}, \mathbb{R}^n)$:

$$\|A_{i_\Omega}^{\beta}\left(e^{-tA_h} - e^{-tA_{i_\Omega}}\right)\| \leqslant C(\epsilon)\frac{1}{t^{\beta}}e^{-bt}, \qquad 0 \leqslant \beta \leqslant 1$$

for $t > 0$, where $b \in \mathbb{R}$ can be chosen as close to a as needed and $C(\epsilon) \to 0$ as $\epsilon \to 0$.

- Moreover, we get the convergence of the resolvent operator $(\xi + A_h)^{-1}$. Indeed, for every $\xi_0 \in \rho(A_0)$, there exists $\delta > 0$ such that

$$\|(\xi + A_h)^{-1} - (\xi + A_{i_\Omega})^{-1}\|_{\mathscr{L}(H^{-1},H^1)} \to 0 \quad \text{as } \epsilon \to 0,$$

whenever $\xi \in \rho(A_{i_\Omega})$ and $|\xi - \xi_0| < \delta$.

- These parabolic problems determine a gradient nonlinear semigroup

$$T^{\epsilon}(t)\phi = u_{\epsilon}(t, \phi), \text{ for } t > 0 \text{ and } \phi \in H^r(\Omega),\ 1/2 \leq r < 1.$$

Since $h^*\Delta_{h(\Omega)}h^{*-1}$ is analytic in h the hypotheses $(H2)$ also holds in $h(\Omega)$ if h is C^2-close enough to the inclusion of $i : \Omega \hookrightarrow \mathbb{R}^n$.

Moreover, under conditions $(H1)$ and $(H2)$, they have a global attractor $\hat{\mathcal{A}}_h$ uniformly bounded in $L^{\infty}(\Omega)$.

Suitable nonlinearities H_h.

Indeed, we have the following result:

Lemma

The application $H : H^r(\Omega) \times \mathrm{Diff}^2(\Omega) \to H^{-1}(\Omega)$, with $\frac{1}{2} < r \leq 1$, given by $H(u,h) = F(u,h) + G(u,h)$ with F and G previously defined, is locally Lipschitz continuous in h uniformly for u in $H^r(\Omega)$, Lipschitz continuous and continuously Fréchet differentiable in u.

The *crux*:

$$\langle G_h(u), \phi \rangle_{-1,1} = \int_{\partial\Omega} \Gamma(g(u))\, \Gamma(\phi) \left| \frac{J_{\partial\Omega} h}{Jh} \right| d\sigma(x).$$

Then we get the continuity of the nonlinear semigroup and the *upper* semicontinuity of the family of attractors $\{\mathcal{A}_h\}_{h \in \mathrm{Diff}^2(\cdot)}$:

Theorem

The family of global attractors $\mathcal{A}_h$ of (2) is upper semicontinuous in $H^r(\Omega)$, with $\frac{1}{2} < r < 1$, at $h = i_\Omega$.

Now, due to the gradient structure of the flow, its attractor is the unstable manifold of the set of equilibria $\mathscr{E}_h$, given by:

$$\mathscr{E}_h = \{\phi \in H^1(\Omega) \, : \, A_h\phi + H_h(\phi) = 0\}.$$

In particular, we must have $\mathscr{E}_h \neq \emptyset$. Also, it follows from the regularization properties of the elliptic operator A_h that $\mathscr{E}_h$ is a compact subset of $H^1(\Omega)$.

Lemma

The equilibria of problem (2) are all hyperbolic for h in a open dense set $\mathcal{H}$ of $h \in \mathrm{Diff}^2(\Omega)$. If $h_0 \in \mathcal{H}$, then the equilibria vary continuously in a neighborhood of h_0.

- The openness and density with respect to boundary perturbations follow from [D. Henry, Cambridge Univ. Press (2005)].
- To prove continuity, we may, without loss of generality, take $h = i_\Omega$, and apply the Implicit Function Theorem from [Loomis, Sternberg, (1968)] for

$$\begin{aligned} F : H^1(\Omega) \times \mathcal{H} &\mapsto H^{-1}(\Omega) \\ (u, h) &\to A_h u + H_h(u), \end{aligned}$$

where $\mathcal{H}$ is a neighborhood of i_Ω.

Finally we have:

Theorem

The family of global attractors $\mathcal{A}_h$ of (2) is lower semicontinuous in $H^r(\Omega)$, with $\frac{1}{2} < r < 1$, at $h = i_\Omega$.

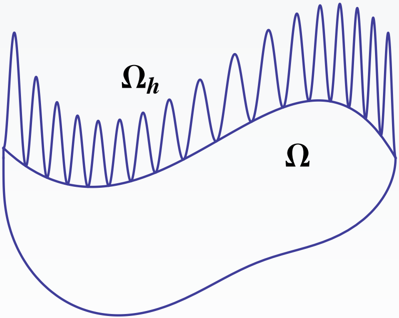
Corollary

The family of global attractors $\mathcal{A}_h$ of (2) is continuous at $h = i_\Omega$ in the topology of C^δ for any $0 < \delta < 1$.

Here we have investigated the continuity of the dynamical structures of parabolic equations with nonlinear boundary conditions assuming h is approaching i_Ω in a $C^2(\Omega)$ topology.

Future works.

We also can consider the case $h \to i_\Omega$ in weak topologies.



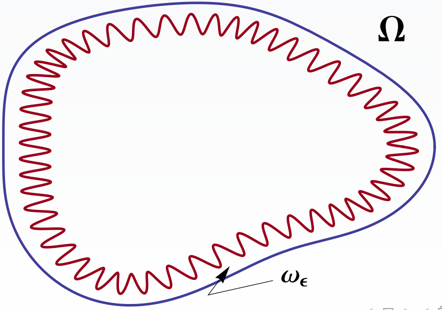
Indeed, we would like to extend the results obtained in **[Arrieta, Bruschi, Math. Models Methods Appl. Sci. (2007)]** and **[Arrieta, Bruschi, Discrete Contin. Dyn. Syst. (2010)]** to parabolic problems.

Next example: Let us consider the asymptotic behavior of the following problem as $\epsilon > 0$ goes to zero:

$$(P_\epsilon) \qquad \begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon + \lambda u_\epsilon + \frac{1}{\epsilon}\mathcal{X}_{\omega_\epsilon} V_\epsilon u_\epsilon = \frac{1}{\epsilon}\mathcal{X}_{\omega_\epsilon} f(u_\epsilon) & \text{in } \Omega, \\ \partial_N u_\epsilon = 0 & \text{on } \partial\Omega, \\ u_\epsilon(0) = \phi_\epsilon \in H^1(\Omega), \end{cases} \quad t > 0.$$

Here the characteristic functions $\mathcal{X}_{\omega_\epsilon}$ models the concentration in the narrow strip $\omega_\epsilon \subset \bar{\Omega}$ through the term

$$\frac{1}{\epsilon}\mathcal{X}_{\omega_\epsilon} \in L^\infty(\Omega).$$



Required conditions:

About the narrow strip

- $\mathcal{X}_{\omega_\epsilon}$ is the characteristic function of the narrow strip $\omega_\epsilon \subset \Omega$

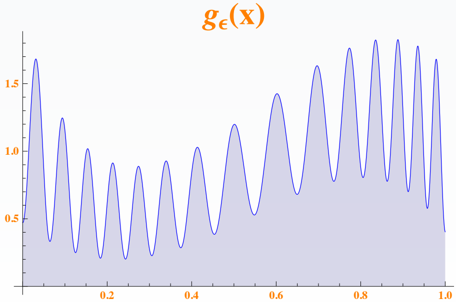
$$\omega_\epsilon = \left\{\xi \in \mathbb{R}^2 \ : \ \xi = \zeta(s) - t\,N(\zeta(s)),\ s \in [0, T] \text{ and } 0 \leqslant t < \epsilon\, g_\epsilon(s)\right\},$$

  $\zeta = (x, y)$ is a C^2-parametrization of $\partial\Omega$, $\|\zeta'(s)\| = 1$, and $N(\zeta) = (y', -x')$ is the unit outward normal vector to $\partial\Omega$.

- $g_\epsilon(\cdot)$ satisfies $0 < g_0 \leqslant g_\epsilon(\cdot) \leqslant g_1$, for g_0 and $g_1 > 0$, such that

$$g_\epsilon(s) = g(s, s/\epsilon),$$

  with $g : (0, T) \times \mathbb{R} \mapsto \mathbb{R}^+$, smooth, $l(x)$-periodic in y, $0 < l_0 < l(\cdot) < l_1$.



About the nonlinearity

- λ is a suitable real number and $f : \mathbb{R} \mapsto \mathbb{R}$ is a C^2-function satisfying the dissipative and growth condition

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0, \qquad |f(u)| + \left|f'(u)\right| + \left|f''(u)\right| \leqslant K,$$

for some constant $K > 0$.

About the potential term

- We assume there exists $C > 0$ such that $V_\epsilon \in L^\infty(\Omega)$ satisfies

$$\frac{1}{\epsilon} \int_{\omega_\epsilon} |V_\epsilon(x, y)|^2 \, dxdy \leqslant C, \quad \forall \epsilon > 0.$$

- Also, we suppose there exists $V_0 \in L^2(\partial\Omega)$ such that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} V_\epsilon \, \varphi \, d\xi = \int_{\partial\Omega} V_0 \, \varphi \, dS, \qquad \forall \, \varphi \in C^\infty(\bar{\Omega}).$$

Roughly, we are assuming that the reactions of the problem (P_ϵ) occur only in an extremely oscillating thin region near the border.

In some sense, we prove the dynamics of this singular problem can be approximated by that one from a parabolic problem with nonlinear boundary conditions, where the oscillatory behavior of the ω_ϵ-strip is captured as a flux condition and a potential term on the boundary.

We show that the limit problem of (P_ϵ) is the following parabolic problem with nonlinear boundary conditions

$$(P_0) \qquad \begin{cases} \partial_t u_0 - \Delta u_0 + \lambda u_0 = 0 \text{ in } \Omega, \\ \partial_N u_0 + V_0 u_0 = \mu f(u_0) \text{ on } \partial\Omega, \; t > 0 \\ u_0(0) = \phi_0 \in H^1(\Omega), \end{cases}$$

where the boundary coefficient $\mu \in L^\infty(\partial\Omega)$ is related to the oscillating function g_ϵ and is given by

$$\mu(s) = \mu(\zeta(s)) = \frac{1}{l(s)} \int_0^{l(s)} g(s, \tau)\, d\tau, \qquad \forall\, s \in (0, T).$$

1. Original problem in H^1.

$$\int_\Omega \partial_t u_\epsilon \varphi + \int_\Omega \nabla u_\epsilon \cdot \nabla \varphi \, d\xi + \lambda \int_\Omega u_\epsilon \varphi \, d\xi + \frac{1}{\epsilon} \int_{\omega_\epsilon} V_\epsilon \, u_\epsilon \, \varphi \, d\xi = \frac{1}{\epsilon} \int_{\omega_\epsilon} f(u_\epsilon) \, \varphi \, d\xi \qquad \text{if } \epsilon > 0.$$

2. Limit problem in H^1.

$$\int_\Omega \partial_t u_0 \varphi + \int_\Omega \nabla u_0 \cdot \nabla \varphi \, d\xi + \lambda \int_\Omega u_0 \varphi \, d\xi + \int_{\partial\Omega} V_0 \, \gamma(u_0) \, \gamma(\varphi) \, d\xi = \int_{\partial\Omega} \mu \, \gamma(f(u_0)) \, \gamma(\varphi) \, d\xi \ \text{ if } \epsilon = 0.$$ [2]

We have to be able to describe how different *concentrated integrals* converge to *surface integrals* to get the limit equation.

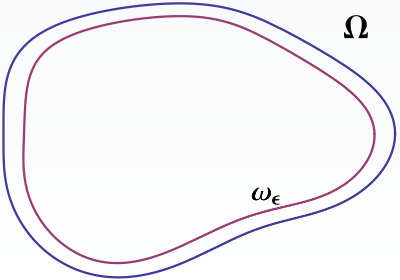
Indeed, if $h, \varphi \in H^s(\Omega)$, with $\frac{1}{2} < s \leqslant 1$, we prove

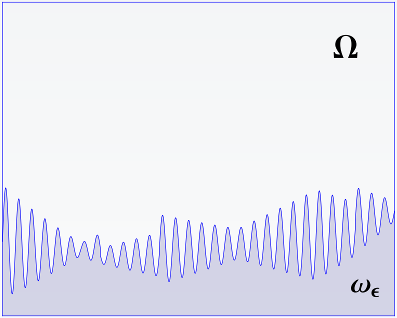
$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} h \, \varphi \, d\xi = \int_{\partial\Omega} \mu \, \gamma(h) \gamma(\varphi) dS.$$

[2] γ denotes the trace operator.

1. In **[Arrieta, Jiménez-Casas, Rodríguez-Bernal, Revista Matemática Iberoamericana 24 (2008)]**, the authors have described how different *concentrated integrals* converge to *surface integrals* without oscillation.
2. In **[Jiménez-Casas, Rodríguez-Bernal, Nonlinear Analysis: Theory, Methods & Applications, (2009)]** and [**A. Jiménez-Casas and A. Rodríguez-Bernal, JMAA (2011)]** the authors also consider the dynamics defined by parabolic problems and obtain *upper semicontinuity* of the attractors.

$$g_\epsilon(s) = k(s) \quad \text{gives us} \quad \mu(s) = k(s).$$





$$g_\epsilon(s) = g(s, s/\epsilon) \quad \text{gives us} \quad \mu(s) = \frac{1}{l(s)} \int_0^{l(s)} g(s,x)\,dx.$$

In **[Aragão, A. Pereira, M. Pereira, Math. Methods in the Appl. Sciences (2012)]** we get the limit problem for a nonlinear elliptic equation with a nonlinear boundary condition capturing the oscillatory behavior of ω_ϵ due to $\mu(s)$, the mean value of $g(s,\cdot)$.

Continuity of attractors[a]

[a] **G. S. Aragão, A. L. Pereira and M. C. Pereira** *Attractors for a nonlinear parabolic problem with terms concentrating in the boundary*. Submitted. arXiv:1204.0117

The goal of this example is:

- To extend the results from Jiménez-Casas and Rodríguez-Bernal to a parabolic problem in which ω_ϵ presents a highly oscillatory behavior.
- Moreover, assuming hyperbolicity of the equilibria of the limit problem, we also obtain results on the lower semicontinuity of the attractors.

To do so, we have to estimate uniformly in ϵ:

1. Linear operators.
2. Nonlinearities and their derivatives.

Working in

- A gradient system.

Assuming

- Each equilibrium of the limit problem is hyperbolic.

Thank you.