Stability results for impulsive functional differential equations with infinite delay and applications

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Introduction

In this work, we consider a system of impulsive differential equations with infinite delay given in abstract form as

$$\dot{x}_i(t) = f_i(t, x_t),\ 0 \leq t \neq t_k, \tag{1}$$

$$\Delta(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k^-)), \quad i = 1, 2, \ldots n,\ k \in \mathbb{N}. \tag{2}$$

Here, $f_i(t, \varphi)$ are real continuous functions for $t \geq 0$ and $\varphi : (-\infty, 0] \to \mathbb{R}^n$ on some space of functions to be defined later, $I_{ik} : \mathbb{R} \to \mathbb{R}$ are continuous, and $\{t_k\}_{k\in\mathbb{N}}$ is a sequence in $(0, \infty)$ such that $t_k \to \infty$ as $k \to \infty$.

For $t \geq 0$, we define $x_t : (-\infty, 0] \to \mathbb{R}^n$ by

$$x_t(s) = x(t+s), \quad s \in (-\infty, 0].$$

To give an initial condition for (1)- (2) at time $t = \sigma$ is to give the past of the system for $s \leq \sigma$, i.e., to require that $x(\sigma + s) = \varphi(s)$ for $s \leq 0$, for some prescribed function $\varphi : (-\infty, 0] \to \mathbb{R}^n$. With the above notation, we write

$$x_\sigma = \varphi. \tag{3}$$

For a compact interval $[\alpha, \beta]$ of $\mathbb{R}$, let $PC([\alpha, \beta]; \mathbb{R}^n)$ be the space of piecewise continuous functions from $[\alpha, \beta]$ to $\mathbb{R}^n$ and left continuous on $(\alpha, \beta]$, $PC([\alpha, \beta]; \mathbb{R}^n) = \{\phi : [\alpha, \beta] \to \mathbb{R}^n | \ \phi$ is continuous everywhere except for a finite number of points $s \in [\alpha, \beta)$ for which $\phi(s^-), \phi(s^+)$ exist and $\phi(s^-) = \phi(s)\}$, equipped with the supremum norm

$$\|\phi\|_\infty = \sup_{s \in [\alpha, \beta]} |\phi(s)|,$$

where $|\cdot|$ is a chosen norm in $\mathbb{R}^n$.

Denote by $R([\alpha,\beta];\mathbb{R}^n)$ the closure of $PC([\alpha,\beta];\mathbb{R}^n)$ with respect to the supremum norm in the space of all bounded functions from $[\alpha,\beta]$ to $\mathbb{R}^n$.

The space $R([\alpha,\beta];\mathbb{R}^n)$ is the space of normalized (from the left) regulated (or ruled) functions from $[\alpha,\beta]$ to $\mathbb{R}^n$, i.e, the space of functions $f:[\alpha,\beta]\to\mathbb{R}^n$ with only discontinuities of the first kind, and left continuous on $(\alpha,\beta]$.

The space $R([\alpha,\beta];\mathbb{R}^n)$ is a Banach space and every function in $R([\alpha,\beta];\mathbb{R}^n)$ has at most countably many discontinuities (see e.g. [Dieudonné, p. 146]).

Define the space $PC = PC((-\infty, 0]; \mathbb{R}^n)$ as the space of functions from $(-\infty, 0]$ to $\mathbb{R}^n$ for which the restriction to each compact interval $[\alpha, \beta] \subset (-\infty, 0]$ is in $R([\alpha, \beta]; \mathbb{R}^n)$.

Clearly, if $\phi \in PC$ then ϕ is continuous everywhere except at most for a enumerable number of points $s = s_k$, and $\phi(s_k^-), \phi(s_k^+)$ exist with $\phi(s_k) = \phi(s_k^-)$.

Denote by BPC the subspace of all bounded functions in PC, $BPC = BPC((-\infty, 0]; \mathbb{R}^n) = \{\phi \in PC : \phi \text{ is bounded on } (-\infty, 0]\}$, with the supremum norm $\|\phi\|_\infty = \sup_{\theta \leq 0} |\phi(\theta)|$. For $\beta \in \mathbb{R}$, in a similar way we define the spaces $PC((-\infty, \beta]; \mathbb{R}^n)$ and $BPC((-\infty, \beta]; \mathbb{R}^n)$.

Fix a function g such that:

(g1) $g : (-\infty, 0] \to [1, \infty)$ is a non-increasing continuous function and $g(0) = 1$;
(g2) $\lim_{u \to 0^-} \dfrac{g(s+u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;
(g3) $g(s) \to \infty$ as $s \to -\infty$.

Define $PC_g = \{\phi \in PC : \ \sup_{s \leq 0} \dfrac{|\phi(s)|}{g(s)} < \infty\}$.

Definition

A function $x(t)$ is called a **solution** *of system (1)-(2) corresponding to the initial data $(\sigma, \phi) \in \mathbb{R} \times BPC$ if there is $d > \sigma$ such that $x : (-\infty, d] \to \mathbb{R}^n$ is continuous for $t \in [\sigma, d] \setminus \{t_k : k \in \mathbf{N}\}$, $x(t_k^-)$ and $x(t_k^+)$ exist with $x(t_k) = x(t_k^-)$ for $t_k \in [\sigma, d]$ $(k \in \mathbf{N})$, the left-derivative of $x(t)$ exists for $t \in [\sigma, d] \setminus \{t_k : k \in \mathbf{N}\}$, $x(t)$ satisfies system (1), and $x_\sigma = \phi$.*

In order to simplify the notation, in general we shall take f in (1) defined in the whole space (i.e, either $D = PC_g$ or $D = BPC$) and initial conditions will be given at $t = 0$:

$$x_0 = \phi, \quad \phi \in BPC. \tag{4}$$

The space PC_g

For differential equations with impulses and unbounded delay, one should take some care with the choice of a suitable phase space and set of initial conditions. Even for the case without impulses, dealing with FDEs with *unbounded* delay requires a careful abstract formulation of an admissible phase space (cf. [Hino, Haddock, etc]). For g satisfying (g1)-(g3), the space UC_g defined by

$$UC_g = \left\{ \varphi \in C((-\infty,0],\mathbb{R}^n) : \sup_{\theta \leq 0} \frac{\varphi(\theta)}{g(\theta)} < \infty, \frac{\varphi(\theta)}{g(\theta)} \text{ is uniformly continuous on } (-\infty,0] \right\}$$

with the norm $\|\cdot\|_g$ is an *admissible* phase space in the sense [Hino, Haddock, etc].

Consider the phase space $\mathcal{D}$ with norm $\|\cdot\|$, where either $(\mathcal{D}, \|\cdot\|) = (PC_g, \|\cdot\|_g)$ for some g satisfying (g1)-(g3), or $(\mathcal{D}, \|\cdot\|) = (BPC, \|\cdot\|_\infty)$, and take BPC as the set of admissible initial conditions.

Suppose that x^* is an equilibrium of (1). The equilibrium x^* is said to be:

- **stable** if for any $\sigma > 0$ and $\varepsilon > 0$ there is $\delta = \delta(\sigma, \varepsilon) > 0$ such that $\|x_t(\sigma, \phi)\| < \varepsilon$ for all $\phi \in BPC$, with $\|\phi\| < \delta$ and $t \geq \sigma$;
- **uniformly stable** if δ in (i) does not depend on σ;
- **globally asymptotically stable** if x^* is stable and globally attractive in $\mathbb{R}^n$, i.e., $x(t) \to x^*$ as $t \to \infty$, for all solutions $x(t)$ with initial condition in BPC.
- **globally exponentially stable** if there are positive constants ε, M such that

$$|x(t, 0, \phi) - x^*| \leq M e^{-\varepsilon t} \|\phi - x^*\|_\infty \quad \forall t \geq 0, \ \phi \in BPC.$$

Lemma

Suppose that $\mathbb{R}^n$ is equipped with the maximum norm, and assume the following hypotheses:

(H1) *for all $t \geq 0$ and $\varphi \in PC_g$ such that $\frac{1}{g(\theta)}|\varphi(\theta)| < |\varphi(0)|$, for all $\theta < 0$, then $f_i(t,\varphi)\varphi_i(0) < 0$ for some $i \in \{1, ..., n\}$ such that $|\varphi(0)| = |\varphi_i(0)|$;*

(H2) *for each function $\hat{I}_k(u) := I_k(u) + u$ $(u \in \mathbb{R}^n)$, there is $\xi_k > 0$ such that $|\hat{I}_k(u)| \leq \xi_k |u|$ and $\prod_{k=1}^{\infty} \max\{1, \xi_k\} < \infty$.*

If $x(t,0,\varphi)$ is a solution of (1)-(2)-(3) defined on $\mathbb{R}$, then $x(t,0,\varphi)$ is bounded and

$$|x(t,0,\varphi)| \leq \|\varphi\|_g \prod_{k=1}^{\infty} \max\{1, \xi_k\}, \quad \forall t \geq 0.$$

Now we consider the following non-autonomous impulsive system:

$$\begin{cases} \dot{x}_i(t) = -a_i(x_i(t))[b_i(x_i(t)) + f_i(t, x_t)], \ 0 \leq t \neq t_k, \ i = 1, 2, \ldots n \\ \Delta(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k^-)), \end{cases} \tag{5}$$

where $a_i : \mathbb{R} \to (0, \infty)$, $b_i : \mathbb{R} \to \mathbb{R}$, $f_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ and $I_{ik} : \mathbb{R} \to \mathbb{R}$ are continuous functions for all $k \in \mathbb{N}, 1 \leq i \leq n$, and either $\mathcal{D} = PC_g$ or $\mathcal{D} = BPC$.

We shall also consider the continuous version of (5),

$$\dot{x}_i(t) = -a_i(x_i(t))[b_i(x_i(t)) + f_i(t, x_t)], \quad t \geq 0, \ i = 1, 2, \ldots n \tag{6}$$

Global Exponential Stability

In the sequel, we fix the maximum norm in $\mathbb{R}^n$, $|x| = \max_{1\le i\le n} |x_i|$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and the following hypotheses will be considered:

- **(A1)** there exist constants $\beta_i > 0$ such that $\dfrac{b_i(u) - b_i(v)}{u - v} \ge \beta_i > 0$, for all $u,\ v \in \mathbb{R}, u \neq v,\ i = 1, 2, \dots, n$;

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- **(A2)** $f_i : \mathbb{R} \times PC_g \to \mathbb{R}$ are uniformly Lipschitz continuous with respect to $\varphi \in PC_g$, with $|f_i(t, \varphi) - f_i(t, \psi)| \le l_i \|\varphi - \psi\|_g$, for $t \in \mathbb{R}$ and φ, $\psi \in PC_g$, $i = 1, 2, \dots, n$;

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- **(A3)** $\beta_i > l_i$, $i = 1, 2, \ldots, n$;

- **(A4)** $\hat{I}_{ik}$ are Lipschitz continuous, with $|\hat{I}_{ik}(u) - \hat{I}_{ik}(v)| \le \hat{\gamma}_{ik}|u - v|$ for $u, v \in \mathbb{R}$, $i = 1, \dots, n$, $k \in \mathbb{N}$, where $\hat{I}_{ik}(u) = u + I_{ik}(u), u \in \mathbb{R}$;

Proposition

Assume (A1)-(A4). Then the initial value problem (4)-(5) *has a solution* $x(t)$ *defined on* $[0, \infty)$.

Lemma

Assume (A1), (A2) and (A3), and that for each $x \in \mathbb{R}^n$ *the functions* $t \mapsto f_i(t, x)$ *are constant on* $\mathbb{R}$, $1 \le i \le n$. *Then there exists a unique equilibrium point* x^* *of* (6).

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- **(A5)** for $\hat{\gamma}_k := \max_{1 \leq i \leq n} \hat{\gamma}_{ik}$, $\prod_{k=1}^{\infty} \max\{1, \hat{\gamma}_k\} < \infty$;

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- **(A5)** for $\hat{\gamma}_k := \max_{1 \leq i \leq n} \hat{\gamma}_{ik}$, $\prod_{k=1}^{\infty} \max\{1, \hat{\gamma}_k\} < \infty$;
- **(A6)** $\inf_{k \geq 1}(t_{k+1} - t_{k-1}) := \sigma > 0$.

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We shall study the global exponential stability of an equilibrium point of the impulsive system (5). For this purpose, we consider the phase space PC_g, with $g(\theta) = e^{-\alpha s}$, $s \in (-\infty, 0]$ for some $\alpha > 0$, and denote PC_g by PC_α and $\|.\|_g$ by $\|.\|_\alpha$.

Lemma

Assume (A1), (A2), (A3), and
(A7) $a_i(u) \geq \underline{a_i} > 0$, *for all* $u \in \mathbb{R}$, $i = 1, 2, \ldots, n$.
Suppose also x^* *is the unique equilibrium point of (6). Consider the solution* $x : (-\infty, b] \to \mathbb{R}^n$ *of the continuous system (6) on* $[a, b]$ *with* $x_a \in PC_\alpha$ $(-\infty < a < b \leq \infty)$*. If there exist constants* $q > 0$ *and* $\varepsilon \in (0, \alpha]$ *with* $\varepsilon < \min_i\{\underline{a_i}(\beta_i - l_i)\}$*, such that*

$$|x(t) - x^*| \leq qe^{-\varepsilon(t-a)} \quad \text{for} \quad t \in (-\infty, a]$$

then

$$|x(t) - x^*| \leq qe^{-\varepsilon(t-a)} \quad \text{for} \quad t \in (-\infty, b) \tag{7}$$

Proof: After the change $y(t) = x(t) - x^*$, we may assume that the equilibrium point is zero, i.e., we consider (6) subject to the constraints $b_i(0) + f_i(t, 0) = 0$, $t \in \mathbb{R}$, $1 \le i \le n$.
By contradiction, suppose that (7) does not hold. Consequently there are $\delta > 0$, $m \in \{1, 2, ..., n\}$, and $t^* \in (a, b]$ such that

$$|y_m(t^*)| = (q + \delta)e^{-\varepsilon(t^* - a)} \quad \text{and} \quad |y_i(t)| < (q + \delta)e^{-\varepsilon(t-a)}, \ \ t < t^*,$$

for $i = 1, ..., n$. Assume that $y_m(t^*) > 0$ (the situation $y_m(t^*) < 0$ is analogous). Denoting $z(t) := (q + \delta)e^{-\varepsilon(t-a)}$, $t \in [a, b]$, we have

$$y_m'(t^*) \ge z'(t^*).$$

On the other hand, using the hypotheses (A1)-(A3) and (A7), we have

$$
\begin{aligned}
y_m'(t^*) = -a_m(y_m(t^*))[b_m(y_m(t^*)) + f_m(t^*, y_{t^*})] = \\
-a_m(y_m(t^*))[b_m(y_m(t^*)) - b_m(0) + f_m(t^*, y_{t^*}) - f_m(t^*, 0)] \\
\leq -\underline{a_m}[\beta_m(y_m(t^*)) - l_m\|y_{t^*}\|_\alpha] \\
\leq -\underline{a_m}[\beta_m z(t^*) - l_m sup_{s\leq 0}\frac{(q+\delta)e^{-\varepsilon(t^*+s-a)}}{e^{-\alpha s}}] \\
\leq -\underline{a_m}[\beta_m z(t^*) - l_m(q+\delta)e^{-\varepsilon(t^*-a)}] = \\
-\underline{a_m}[(\beta_m - l_m)z(t^*) < -\varepsilon z(t^*) = z'(t),
\end{aligned}
$$

which is a contradiction.

Main Result

Theorem

Assume that there is an equilibrium x^* of (5). Assume also (A1)–(A4) and (A7) and
(A8) *for $\hat{\gamma}_k = \max_{1 \le i \le n} \hat{\gamma}_{ik}$ as in (A4) and some $k_0 \in N$,*

$$\eta := \sup_{k \ge k_0} \left(\frac{\log(\max\{1, \hat{\gamma}_k\})}{t_k - t_{k-1}} \right) < \min_i \{\underline{a_i}(\beta_i - l_i)\} \tag{8}$$

where the space PC_g in (A2) is $PC_g = PC_\varepsilon$ for some $\varepsilon > \eta$. Then the equilibrium x^* of (5) is globally exponentially stable.

Applications

We apply the previous results to the following generalized Cohen-Grossberg neural network model with both time-varying delays and distributed delays:

$$\dot{x}_i = -a_i(x_i(t))[b_i(x_i(t)) - \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij}(t))) - \sum_{j=1}^{n} c_{ij} h_j(\int_{-\infty}^{0} k_{ij}(-s) x_j(t+s) ds) + J_i], \quad 0 \le t \ne t_k \quad (9)$$

$$\Delta(x_i(t_k)) = I_{ik}(x_i(t_k^-)), \quad i = 1, ..., n, k \in N,$$

where $a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$, $a_i : \mathbb{R} \to (0, \infty)$ and $\tau_{ij} : [0, \infty) \to [0, \infty)$ are continuous functions, with $\tau_{ij}(t) \leq \tau_{ij} \leq \tau$, f_i, g_i, h_i are Lipschitz functions with Lipschitz constants F_j, G_j and H_j respectively, and K_{ij} are nonnegative continuous functions such that

$$\int_0^\infty k_{ij}(t)dt = 1, \quad i, j = 1, ..., n, k \in N.$$

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Thank you for your attention!