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# Envelope Solutions to PDEs Depending Of Two Disjoint Sets of Variables

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**Abstract**

There are a lot of applications for the envelope solutions to PDEs, the hypersurfaces that enclose one of the families of the hypersurfaces given by the complete solutions. The development and discussion of the existence of envelope solutions to PDEs that depends of two disjoint sets of variables are the main purpose of this research. As an example it is considered the canonical variables describing a mechanical system at the phase space in Hamiltonian Analytical Mechanics. As one of the possible extensions it will be discussed the development and the analyses of the existence of envelope solutions to the variational PDEs that involves functional depending of two disjoint sets of variables. As it occurs in Hamiltonian Analytical Mechanics applied to field theories where the dependence is of the field functions and the canonical variables represented by the density momenta.

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# 1 Introduction

The envelope solution has an important role in PDEs and in ODEs, mainly in applied mathematics. In the theory of PDEs the envelope solutions are the hypersurfaces that enclose one of the families of the hypersurfaces given by the complete solutions. The development and discussion of the existence of envelope solutions to PDEs that depends of two disjoint sets of variables are the main purpose of this paper. These results where so developed to justify the results used in the development of the alternative Hamiltonization procedure in the foundations of analytical mechanics. In Hamiltonian analytical mechanics the functions are defined in the phase space depending therefore of two disjoint sets composed by a set of generalized coordinates and another set of the conjugate momenta[1]. A procedure named alternative Hamiltonization developed by [2] shows that the usual Hamiltonian function defined by Hamilton is the envelope solution of a partial differential equation obtained by the use of the first set of canonical equation of motion in the definition $H(p,q) = p\dot{q} - L(\dot{q},q)$, where $L(\dot{q},q)$ is the Lagrangian function and $H(p,q)$ the Hamiltonian one. In Hamiltonian procedure one passes from the configuration space $(\dot{q},q)$ to the phase space $(p,q)$ defining the generalized momenta as $p = \partial L/\partial\dot{q}$. The alternative Hamiltonization procedure proves that this definition implies in the choice of a Hamiltonian function given by the envelope solution to the above PDE. The existence of the envelope solution is discussed in the scope of these application. The result obtained for the singular mechanics, implies an equation that do not have an envelope solution. In the alternative Hamiltonization procedure these results are used to prove that there are no envelope solution if the partial differential equation that defines the Hamiltonian function is linear in the derivative, and despite of the use of this results there are nowhere in the mathematical literature this prove. This article intend to do this extension of the usual condition to obtain the envelope solution, and also discuss the existence of these solutions.

# 2 Envelope Solutions to PDEs Depending Of Two Disjoint Sets of Variables

As the main purpose of the discussion about the envelope solutions and their existence are in Hamiltonian mechanics it will be considered two disjoint sets of variables. The phase space $(p,q)$ is composed by the set of generalized variables $q$ and the one of the generalized momenta $p$.

Let consider the two sets as $x = x_1, ..., x_n$, $y = y_1, ..., y_n$ and a function $u = u(x,y)$. The partial differential equation of first order to be considered is given by

$$u(x,y) = f(p,q,x,y), \tag{1}$$

where $p = p_1, ..., p_n$, $p = \partial u/\partial x$ and $q = q_1, ..., q_n$, $q = \partial u/\partial y$. The general solutions can be written as

$$\varphi(u, x, y) = 0, \tag{2}$$

a corresponding complete solution family can be obtained from

$$\varphi(u, x, y, a, b) = 0, \tag{3}$$

where $a = a_1, ..., a_n$ and $b = b_1, ..., b_n$ are two sets of constants. The imposition of the envelope condition[3], results in

$$\frac{\partial \varphi}{\partial a_i} = 0, \qquad \frac{\partial \varphi}{\partial b_i} = 0, \tag{4}$$

Then the solution of the system obtained from equations (3) and (4) gives the envelope solution.

To give a more useful form to the application suppose that that solution of the PDE can be written in the explicit manner as

$$u(x, y) = \phi(x, y), \tag{5}$$

then the complete solution is given by

$$u(x, y) = \phi(x, y, a, b). \tag{6}$$

And the envelope conditions given by equations (4) are

$$\frac{\partial \phi}{\partial a_i} = 0, \qquad \frac{\partial \phi}{\partial b_i} = 0. \tag{7}$$

These system of $2n$ equations gives the sets $a$ and $b$, therefore the envelope solution is obtained from (6).

Furthermore if the equation (1) is linear in $p_i$ (or $q_i$) and also do not depend of $q_i$ (or of $p_i$) then this equation do not have an envelope solution.

As an example let consider the PDE, that has the form similar to the one obtained in singular mechanics

$$u(x, y) = (y_i - r_i(x))\, q_i + s(x),^{1} \tag{8}$$

whose general solution, is given by

$$u = s + \psi_i\, (y_i - r_i), \tag{9}$$

if $y_i \neq r_i$, where $\psi_i = \psi_i(x)$ are arbitrary functions. Then a complete solution can be given by

$$\varphi = s + a_i(y_i - r_i) - u = 0, \tag{10}$$

where $a_i$ are arbitrary constants. As the envelope conditions are imposed by the equation (7), results in a contradiction of the initial hypothesis $y_i - r_i \neq 0$. Therefore the partial differential equation do not have an envelope solution.

[1] Repeated indices mean sum.

# 3 FINAL REMARKS

The absence of an envelope solution in singular mechanics determines a procedure that must differ from that used by Hamilton, as the one developed by Dirac [4] or by Espindola [2].

The results obtained can be extended to N disjoint sets of variables. But the most interesting extension is to field theory. Similar results about the existence of envelope solutions are used in the alternative Hamiltonization procedure developed in field theory [5]. The alternative Hamiltonization procedure, developed by [5], applied to the continuous case uses a similar result as in the mechanical systems, and the lack of an envelope solution implies in the need of a different procedure to obtain an Hamiltonian function . As a consequence of the continuity of the field variables we have variational partial differential equation where the results obtained above can be extended. Then it can be discussed the development and the analysis of the existence of envelope solutions to the variational PDEs that involves functional depending of two disjoint sets of variables composed by the field functions and the conjugate variables represented by the density momenta.

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