Thermodynamic Formalism for one-dimensional spin-lattices: entropy and pressure

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Let (M, d_M) be a compact metric space. We consider the metric in $M^{\mathbb{N}}$:

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_M(x_n, y_n),$$

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For a fixed potential $A \in F_\alpha$ we define a Transfer (Ruelle) operator $\mathcal{L}_A : \mathcal{C} \to \mathcal{C}$ by the rule

$$\mathcal{L}_A(\varphi)(x) = \int_M e^{A(ax)} \varphi(ax) d\nu(a) \,,$$

where $a \in M$, $x \in \mathcal{B}$ and we will use the following notation $ax = (a, x_1, x_2, ...)$, is a pre-image of x.

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$$\bar{A} := A + \log \psi_A - \log \psi_A \circ \sigma - \log \lambda_A.$$

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(b) the measure $\rho_A = \frac{1}{\psi_A}\, \mu_A$ satisfies $\mathcal{L}^*_A(\rho_A) = \lambda_A \rho_A$.
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We call μ_A the **Gibbs state** for A. We will leave the term **equilibrium state** for the one which maximizes pressure.

Definition Consider ν a fixed a-priori probability on M. Denote $\mathcal{G} = \mathcal{G}_\nu$ the set of Gibbs measures, that means, $\mathcal{L}_B^*(\mu) = \mu$ for some normalized potential $B \in F_\alpha$. We define the entropy of $\mu \in \mathcal{G}$ as

$$h(\mu) \,=\, h^\nu(\mu) \,= -\int_{\mathcal{B}} B(x)d\mu(x).$$

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If our a-priori probability is $\nu = \frac{1}{d}\sum_{j=1}^d \delta_j$, to get the above entropy $h(\mu)$ you have just to add $-\log d$ to the classical one $H(\mu)$. Therefore, the above definition of $h(\mu)$ results in a number between $-\log(d)$ and 0.

We point out that at level 2 the Deviation Function for the maximal entropy probability is $I(\mu) = \log d - H(\mu) \geq 0$.
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Proposition If $\mu \in \mathcal{G}$, we have $h(\mu) \leq 0$.

Proof: Let μ be a probability on $\mathcal{G}$ with associated normalized potential B. Then, we have

$$h(\mu) = -\int_{\mathcal{B}} B(x) d\mu(x) = \int_{\mathcal{B}} \log e^{-B(x)} d\mu(x) \leq \log \int_{\mathcal{B}} e^{-B(x)} d\mu(x) =$$

$$= \log \int_{\mathcal{B}} e^{-B(x)} d\mathcal{L}_B^*(\mu)(x) = \log \int_{\mathcal{B}} \mathcal{L}_B e^{-B(x)} d\mu(x) = 0,$$

where we have used Jensen inequality and also $\mathcal{L}_B e^{-B(x)} = 1$.

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The Entropy of a measure supported on a fixed point.

Suppose $M = [0,1]$, and $A_c : M^{\mathbb{N}} \to \mathbb{R}$ is given by $A_c(x) = \log \left(\frac{c}{1-e^{-c}} e^{-cx_1} \right)$.

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Then, we have $h(\mu) \leq -\int A_c d\mu = -A_c(0^\infty) = -\log\left(\frac{c}{1-e^{-c}}\right) \to -\infty$ when $c \to \infty$.

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This shows that $h(\mu) = -\infty$.

Lemma Consider a Holder continuous potential A and $\mu \in \mathcal{G}$ associated to B normalized, and call $\mathcal{C}^+$ the space of continuous bounded positive functions on $\mathcal{B}$. Then,

$$h(\mu) + \int_{\mathcal{B}} A(x) d\mu(x) = \inf_{u \in \mathcal{C}^+} \left\{ \int_{\mathcal{B}} \log \left(\frac{\mathcal{L}_A u(x)}{u(x)} \right) d\mu(x) \right\},$$

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Theorem: Given a Lipchitz potential A consider the associated normalized potential $\overline{A}$, then, the Gibbs probability for $\overline{A}$ is the equilibrium state for A, that is, attains the supremum value of $h(\mu) + \int_{\mathcal{B}} A(x) d\mu(x)$ among invariant probabilities for σ.

The differentiable structure and the involution kernel

We consider now the general XY model. This is the case where $M = S^1$ and the a priori measure is the Lebesgue measure on the circle. $(S^1)^{\mathbb{N}}$ has a differentiable C^∞ structure.

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Let $\mathcal{B}^* = \{y = (....y_2, y_1) \in (S^1)^{\mathbb{N}}\}$, and, we denote by the pair

$$\langle y|x\rangle = \langle ..., y_2, y_1|x_1, x_2...\rangle,$$

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We denote $\hat{\sigma}$ the shift on $\hat{\mathcal{B}}$, and, therefore

$$\hat{\sigma}(\langle ..., y_2, y_1|x_1, x_2, ...\rangle) = \langle ..., y_2, y_1, x_1|x_2, x_3, ...\rangle.$$

Definition Let $A : \mathcal{B} \to \mathbb{R}$ be a continuous potential (considered as a function on $\hat{\mathcal{B}}$). A continuous function $W : \hat{\mathcal{B}} \to \mathbb{R}$ is called a involution kernel if for all $< y, x > \in \hat{\mathcal{B}} := \mathcal{B}^* \times \mathcal{B} = (S^1)^{\mathbb{Z}}$

$$A^* := A \circ \hat{\sigma}^{-1} + W \circ \hat{\sigma}^{-1} - W$$

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Proposition Denote $Z(y|x) = e^{W(y|x)-c}$, then,

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$\hat{\mu}_A = Z(y|x)\, d\rho_{A^*}(y) d\rho_A(x)$ is invariant for $\hat{\sigma}$ and is the natural extension of the Gibbs measure μ_A.

Moreover, $\psi_A(x) = \int_{\mathcal{B}^*} Z(y|x)\, d\rho_{A^*(y)}$ is the main eigenfunction for $\mathcal{L}_A$.

Suppose that A is differentiable in each coordinate of $x \in \mathcal{B}$, and, that given $\epsilon > 0$, there exists $H_\epsilon > 0$, such that, if $|h| < H_\epsilon$, then

$$|\frac{A(x+he_j)-A(x)}{h} - D_jA(x)| \leq \frac{\epsilon}{2^j} \ , \ \forall j \in \mathbb{N},$$

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Proposition Suppose A is differentiable. Then, the eigenfunction ψ_A is differentiable, and

$$\frac{\partial}{\partial x_j}\psi_A(x) = \int_{\mathcal{B}^*} Z(y|x) \sum_{n\geq 1} D_{n+j}A(y_n...y_1 x)\, d\rho_{A^*}(y).$$

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In the case where A depends only on the two first coordinates
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In the case where A depends only on the two first coordinates $\psi_A(x_1) = \frac{1}{\lambda_A}\int_{S^1} e^{A(y_1,x_1)}\psi_A(y_1)d\nu(y_1).$

Hence, ψ_A satisfies the equation $\frac{\partial}{\partial x_1}\psi_A(x_1) = \frac{1}{\lambda_A}\int_{S^1} e^{A(y_1,x_1)} D_2A(y_1,x_1)\psi_A(y_1)d\nu(y_1).$

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We would like to investigate general properties of the limits of $\mu_{\beta_n A}$, and, of $\psi_{\beta_n A}$ when $\beta_n \to \infty$.

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In this case in the XY model the Large Deviation function for $\mu_{\beta A}$, when $\beta \to \infty$ is considered in

[LM] A. Lopes and J. Mengue, Selection of measure and a Large Deviation Principle for the general one-dimensional XY model (2011)

The calibrated subaction plays an essential role on this proof.

Definition A continuous function $u : \mathcal{B} \to \mathbb{R}$ is called a calibrated subaction for $A : \mathcal{B} \to \mathbb{R}$, if, for any $y \in \mathcal{B}$, we have

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Proposition - Existence of calibrated subaction - For any uniformly convergent subsequence $\beta_n \to \infty$, if
$V := \lim_{n\to\infty} \frac{1}{\beta_n} \log(\psi_{\beta_n A})$,
then V is a calibrated subaction for A. In the case we have uniqueness of μ_∞, then selection.

An application to the non-compact case

Suppose $M_0 = \{z_i, i \in \mathbb{N}\}$ is an increasing infinite sequence of points in $[0, 1)$, and suppose $z_\infty := 1 = \lim_{i \to \infty} z_i$. We will assume $z_1 = 0$. Then, $M = M_0 \cup \{1\}$ is a compact set.

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Therefore, we have a state space M_0 that can be identified with $\mathbb{N}$, and M has a special point $z_\infty = 1$ playing the role of the infinity. Let $\mathcal{B}_0 = M_0^{\mathbb{N}}$ and $\mathcal{B} = M^{\mathbb{N}}$.

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Theorem Suppose that $A : \mathcal{B}_0 \to \mathbb{R}$ is a Holder continuous potential. Then it can be extended as a Holder continuous function to $A : \mathcal{B} \to \mathbb{R}$. If the extension satisfies

$A(x_1, ..., x_{n-1}, 1, x_{n+1}, x_{n+2}, ...) < A(x_1, ..., x_{n-1}, 0, x_{n+1}, x_{n+2}, ...)$ for any $n \in \mathbb{N}$ and $x_i \in M$ then:

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a) A has a calibrated subaction V on $\mathcal{B}_0$: for any $x \in \mathcal{B}_0$,

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b) Any maximizing measure for A has support on $\mathcal{B}_0$.

Markov Chains with values on the interval $[0, 1]$

Suppose $M = [0, 1]$.

Let $K : M^2 \to \mathbb{R}$, $\theta : M \to \mathbb{R}$ satisfying

$$\int_M K(x_1, x_2) d\nu(x_2) = 1, \forall x_1 \quad \text{and} \quad \int_M \theta(x_1) K(x_1, x_2) d\nu(x_1) = \theta(x_2) \ , \forall x_2$$

We call K a transition kernel and θ the stationary measure for K.

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We define the Markov measure associated to K and θ as

$$\mu(A_1...A_n \times [0,1]^{\mathbb{N}}) := \int_{A_1...A_n} \theta(x_1)\, K(x_1,x_2)...K(x_{n-1},x_n)\, d\nu(x_n)...d\nu(x_1)$$

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Proposition Given $A(x_1,x_2)$ a continuous potential that depends on two coordinates, there exists a Markov measure that is Gibbs for A.

Proposition Let μ be the Markov measure defined by a transition kernel K and the stationary measure θ. The definition of penalized entropy given in [LMST]:

$$S(\theta K) = -\int_{M^2} \theta(x_1) K(x_1, x_2) \log(K(x_1, x_2)) d\nu(x_1) d\nu(x_2) \leq 0$$

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For the general problem for a metric space M one can consider an a priori probability ν_z on M which depends on $z \in M$ and define an Ruelle operator

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Under some mild assumptions on ν_z, $z \in M$, one can show several of the above properties.

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