Generalized characteristic equations for non-autonomous functional differential equations

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Goal

Goal of the talk

Present the Generalized Characteristic Equation
for nonautonomous FDE,
comparing them with characteristic equations for autonomous FDE.

We use the *asymptotic behavior* as a tool for comparation,
in particular results related to FDE with “small delay”.

History of the results os type "$V < 1$"

- Driver, Sasser & Slater, *The equation* $x'(t) = ax(t) + bx(t - \tau)$ *with "Small" Delay*, AMM (1973).
- Arino & Pituk, *More on linear differential systems with small delays*, JDE (2001).
- Dix, Philos & Purnaras, *Asymptotic properties of solutions to linear non-autonomous neutral differential equations*, J. Math. Anal. Appl. (2006)
  and several works of these authors.
- Frasson & Verduyn Lunel, *Large Time Behaviour of Linear Functional Differential Equations*, IEOT (2003).
- Cuevas & Frasson, *Asymptotic properties of solutions to linear nonautonomous delay differential equations through generalized characteristic equations*, El. J. Diff. Equ. (2010).

Linear autonomous FDE

- $\mathcal{C} = \{\varphi : [-1, 0] \to \mathbb{C} : \varphi \text{ is continuous}\}$
- $x_t \in \mathcal{C}$ defined as $x_t(\theta) = x(t + \theta)$.
- $L : \mathcal{C} \to \mathbb{C}$ bounded linear.

Linear autonomous FDE

$$\dot{x}(t) = Lx_t \tag{FDE}$$

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Example

$\dot{x}(t) = ax(t) + bx(t-1)$ is on form (FDE) when

$$L\varphi = a\varphi(0) + b\varphi(-1)$$

$$Lx_t = ax_t(0) + bx_t(-1) = ax(t) + bx(t-1)$$

Spectral theory for autonomous FDE

- Let $T(t) : \mathcal{C} \to \mathcal{C}$ be the solution operator: $x_t = T(t)\varphi$
- Let $A : \mathcal{C} \to \mathcal{C}$ be the infinitesimal generator of $T(t)$.
- Then

$$\sigma(A) = \{\lambda \in \mathbb{C} : \det \Delta(\lambda) = 0\}$$

Characteristic matriz $\Delta(\lambda)$ of (FDE):

$$\Delta(z) = zI - \int_0^1 d\eta(t) e^{-zt}.$$

and η:

$$L\varphi = \int_0^1 \eta(\theta)\varphi(-\theta).$$

Characteristic equation on the scalar case

Definition

Characteristic equation

$$z - \int_0^1 d\eta(t) e^{-zt} = 0$$

or

$$z - Le^{z\cdot} = 0.$$

One may obtain roots λ of the characteristic equation of a FDE by looking for solutions of the form $x(t) = e^{\lambda t}$.

Spectral projection

- For each eigenvalue λ, we associate a eigenspace

$$\mathcal{M}_\lambda = \ker(\lambda I - A)^k \subset \mathcal{C}, \quad k \text{ “sufficiently large”}.$$

- $\dim \mathcal{M}_\lambda < \infty$
- $\mathcal{M}_\lambda$ is $T(t)$-invariant
- The spectral projection is the projection

$$P_\lambda : \mathcal{C} \to \mathcal{M}_\lambda \text{ along } \mathcal{R}((\lambda I - A)^k)$$

- $\mathcal{Q}_\lambda := (I - P_\lambda)\mathcal{C}$ is $T(t)$-invariant too.
- $\mathcal{C} = \mathcal{M}_\lambda \oplus \mathcal{Q}_\lambda$

Dominant eigenvalues

Definition (Dominant eigenvalue)

$\lambda_d \in \sigma(A)$ is a dominant eigenvalue if $\exists \epsilon > 0$ such that

$$\det \Delta(\lambda) = 0, \quad \lambda \neq \lambda_d \implies \Re\lambda_d > \Re\lambda + \epsilon.$$

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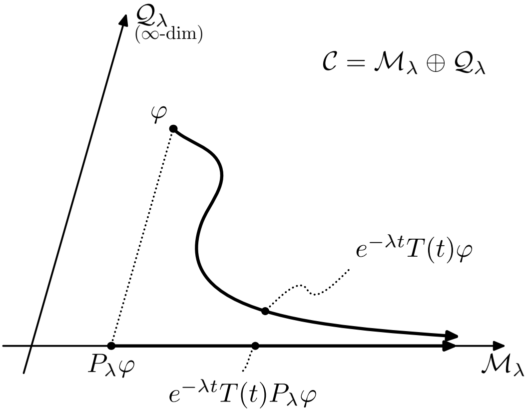
Theorem

Suppose that λ_d is a dominant eigenvalue of A.
Then, for suff. small $\delta > 0$, $\exists K = K(\delta) > 0$ such that

$$\|T(t)(I - P_{\lambda_d})\varphi\| \leq Ke^{(\Re\lambda_d - \delta)t}\|\varphi\|, \quad t \geq 0.$$

Asymptotic behavior for linear autonomous FDE

- Flow of the solution weighted by $e^{-\lambda t}$



A result on dominant eigenvalues

Theorem (Frasson-2009, *Appl. Math. Comput.*)

Let $\lambda \in \mathbb{C}$ be an eigenvalue such that

$$V(\lambda) = \int_0^1 \theta |e^{-\lambda\theta}| d|\eta|(\theta) < 1. \qquad (V < 1)$$

Then λ is a simple dominant eigenvalue.

Non-autonomous linear FDE

- $L(t) : \mathcal{C} \to \mathbb{C}$ a family of linear bounded operators

Non-autonomous linear RFDE

$$\dot{x}(t) = L(t)x_t \qquad \text{(NonAutFDE)}$$

- No standard spectral theory
- No characteristic equation

Generalized characteristic equations

Generalized characteristic equations

- Look for solutions of (NonAutFDE) of the form

$$x_\lambda(t) = \exp\left(\int_0^t \lambda(s)ds\right)$$

- if $L(t)\varphi = \int_0^1 d_\theta \eta(t,\theta)\varphi(-\theta)$

  Generalized characteristic equation:

$$\lambda(t) = \int_0^r d_\theta \eta(t,\theta) \exp\left(-\int_{t-\theta}^t \lambda(s)ds\right) \qquad \text{(GenCharEq)}$$

Asymptotic behavior

Theorem (Cuevas-Frasson-2010 *Electron. J. Diff. Eq.*)

If $\lambda(t)$ is a solution of (GenChaEq) such that

$$\limsup_{t\to\infty} \int_0^r \theta \left| e^{-\int_{t-\theta}^t \lambda(s)ds} \right| d_\theta |\eta|(t,\theta) < 1. \qquad (V<1)$$

Then for each solution x of (NonAutFDE), we have

$$\lim_{t\to\infty} x(t) e^{-\int_{t_0}^t \lambda(s)ds} \text{ exists,}$$

$$\lim_{t\to\infty} \left[x(t) e^{-\int_{t_0}^t \lambda(s)ds} \right]' = 0.$$

Examples

Example (nonautonomous FDE with variable delay)

$$x'(t) = \frac{x(t-\tau(t))}{t+c-\tau(t)}, \quad t \geqslant t_0. \tag{1}$$

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Generalized characteristic equation

$$\lambda(t) = \frac{1}{t+c-\tau(t)} \exp\left[-\int_{t-\tau(t)}^{t} \lambda(s)ds\right] \tag{GChar-1}$$

has solution $\lambda(t) = 1/(t+c)$.

$$(V<1): \quad \limsup_{t\to\infty} \frac{\tau(t)}{t+c} = 0 < 1.$$

$\therefore$ solutions of (1) satisfy $x(t) = O(t)$, $\dot{x}(t) = o(t)$ as $t \to \infty$.

Examples

Example (nonautonomous FDE with distributed delay)

$$x'(t) = \int_0^1 \frac{x(t-\theta)}{t-\theta} d\theta, \quad t > 1. \tag{2}$$

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Generalized characteristic equation

$$\lambda(t) = \int_0^1 \frac{1}{t-\theta} \exp\left[-\int_{t-\theta}^t \lambda(s)ds\right] d\theta \tag{GChar-2}$$

has solution $\lambda = 1/t$.

$$(V < 1): \quad \int_0^1 \frac{\theta}{t} d\theta = \frac{1}{2t} \to 0 \text{ as } t \to \infty.$$

$\therefore$ solutions of (2) satisfy $x(t) = O(t)$, $\dot{x}(t) = o(t)$ as $t \to \infty$.

Final thoughts

- Exponential dichotomy?
- Invariants and asymptotic behavior?

Thank you!