On the uniqueness of degree in infinite dimension

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The Sugarcane Symposium in Real Analysis XXXVII

3-6 de junho de 2013, ICMC – USP São Carlos, SP, Brasil

What is the topological degree?

The degree is a number, associated to an equation

$$f(x) = y, \quad x \in U, \tag{1}$$

in order to obtain information about the set of solutions. In the above equation we can imagine that (for example)

i) $f : X \to Y$ is a given function, supposed at least continuous,

ii a) X and Y are Euclidean spaces or real, finite dimensional, differentiable manifolds or

ii b) Banach spaces or manifolds, possibly of infinite dimension,

iii) y is a fixed element of Y,

iv) U is an open subset of X.

In finite dimension

In $\mathbb{R}^n$ consider the set $\mathcal{T}$ of the *admissible triples*,

$$\mathcal{T} = \{(f, U, y)\},$$

where

i) $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous,

ii) U is an open subset of $\mathbb{R}^n$,

iii) $y \in \mathbb{R}^n$ is such that $f^{-1}(y) \cap U$ is compact.

A *topological degree* is a map

$$\deg : \mathcal{T} \to \mathbb{Z} \quad (\text{or } \deg : \mathcal{T} \to \mathbb{Z}_2)$$

verfying some particular properties.

Such properties give information about $f^{-1}(y) \cap U$.

The degree theory in finite dimension is commonly known as Brouwer degree theory.

L.E.J. Brouwer, *Uber Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1912), pp. 97–115.

M. Nagumo, *A theory of degree of mapping based on infinitesimal analysis*, Amer. J. of Math., **73** (1951), 485–496.

In the particular case of admissible triples (f, U, y) such that f is C^1 and y is a regular value for f in U, the *Brouwer degree* of (f, U, y) is given by

$$\deg_B(f, U, y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sign} Df(x), \tag{2}$$

where $\operatorname{sign} Df(x)$ is the sign of the determinant of the Jacobian matrix associated to $Df(x)$.

Brouwer degree for continuous maps between finite dimensional oriented manifolds.

Let M and N be two oriented smooth real manifolds of the same finite dimension.

In the particular case when $f : M \to N$ is C^1, U is an open subset of M, $y \in N$ is such that $f^{-1}(y) \cap U$ is compact (i.e. we call (f, U, y) *admissible*), we put

$$\deg_B(f, U, y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sign} Df(x), \tag{3}$$

where $\operatorname{sign} Df(x) = 1$ if $Df(x)$ preserves the orientations of the tangent spaces $T_x M$ and $T_{f(x)} N$, and $\operatorname{sign} Df(x) = -1$ otherwise.

In the book of **A. Dold**, *Lectures on algebraic topology,* Springer-Verlag, Berlin, 1972, we find an extension to nonorientable manifolds.

In infinite dimension?

It is not possible to construct a degree theory for (simply) continuous maps between infinite dimensional Banach spaces.

Generally, we have obstructions in the attempt to extend degree to infinite dimension. Let us mention three:

- if $f : E \to F$ is C^1, between Banach spaces, and x is a regular point, it is not clear how we can define a sign for the Fréchet derivative $Df(x)$ and thus it is not clear how to generalize formula (**??**);
- a general result ensuring the approximation of continuous maps by smooth maps does not exist;
- if U is bounded, f is continuous on $\overline{U}$ and y is a given element in F, then $f^{-1}(y) \cap \overline{U}$ is not necessarily compact.

The first (classical) construction:

J. Leray and **J. Schauder**, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup., 51 (1934), 45–78.

Other important contribution of 1936:

R. Caccioppoli, *Sulle corrispondenze funzionali inverse diramate: teoria generale e applicazioni ad alcune equazioni funzionali non lineari e al problema di Plateau*, Opere scelte, vol. II, Edizioni Cremonese, Roma, 1963, 157–177.

The Leray-Schauder degree is defined for maps of the form

$$f : E \to E, \qquad f(x) = x - k(x),$$

where E is a real Banach space, k is *completely continuous*.

The *admissible triples* are those (f, U, y) such that f is as above, y belongs to E and $U \subseteq E$ is open with $f^{-1}(y) \cap U$ compact.

Crucial properties of completely continuous maps:

(P1) Given a completely continuous map $k : E \to E$ and a closed bounded subset B of E, for any $\varepsilon > 0$ there exists a continuous map $k_1 : B \to E$ such that

1. $k_1(B)$ is contained in a finite dimensional subspace of E,
2. $\sup_{x\in B} \|k(x) - k_1(x)\| < \varepsilon$.

(P2) $I - k$ is proper on closed bounded subsets of E.

Given an admissible triple (f, U, y), let D be an open bounded subset of U containing $f^{-1}(y) \cap U$, such that $\overline{D} \subseteq U$.

Write $f(x) = x - k(x)$, with k completely continuous.

Let $k_1 : \overline{D} \to E$ be a continuous approximation of k_1 with image contained in a finite-dimensional subspace E_1 of E.

Define

$$\deg_{LS}(f, U, y) = \deg_B(I - k_1, E_1 \cap D, y). \tag{4}$$

Let us point out that in the Leray–Schauder degree a concept of orientation in infinite dimension is implicitely contained.

Not an orientation of spaces, but an orientation of maps.

In fact, observe that the set $GL_c(E)$ of automorphisms of the form $I-K$, with K linear and compact, has two connected components (while $GL(E)$ could be connected).

H. Amann and **S.A. Weiss**, *On the uniqueness of the topological degree,* Math. Z., 130 (1973), 39–54.

The Brouwer degree and the Leray-Schauder degree are uniquely determined by three fundamental properties, considered as axioms (which we recall in the case of Leray–Schauder degree).

i) (*Normalization*) $\deg_{LS}(I, E, 0) = 1$ (I is the identity of E).

ii) (*Additivity*) Given an admissible triple (f, U, y) and two disjoint open subsets U_1, U_2 of U such that $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$, then, $\deg_{LS}(f, U, y) = \deg_{LS}(f|_{U_1}, U_1, y) + \deg_{LS}(f|_{U_2}, U_2, y)$.

iii) (*Homotopy invariance*) Let $H : U \times [0, 1] \to F$ be of the form $H(x, t) = x - k(x, t)$, with k completely continuous. Let $y : [0, 1] \to F$ be a continuous path. If the set

$$\{(x, t) \in U \times [0, 1] : H(x, t) = y(t)\}$$

is compact, then $\deg_{LS}(H_t, U, y(t))$ in independent of $t \in [0, 1]$.

Extensions of Leray–Schauder degree

S. Smale (1965): nonlinear (C^2) Fredholm maps between Banach space. Degree in $\mathbb{Z}_2$ (with no use of orientation).

F. Browder and **R. Nussbaum** (1969): noncompact perturbations of the identity in a Banach space (using the Kuratowski measure of noncompacness).

K.D. Elworthy and **A.J. Tromba** (1970): *oriented degree* for nonlinear Fredholm maps of index zero between Banach manifolds (introducing the notion of orientation for an infinite dimensional manifold).

J. Mawhin (1972): *Coincidence degree:* for special perturbations of a linear Fredholm operator between Banach spaces.

V.G. Zvyagin and **N.M. Ratiner** (1991): following the concept of orientation of Elworty and Tromba, degree for completely continuous perturbations of nonlinear Fredholm maps of index zero between Banach spaces.

P.M. Fitzpatrick, **J. Pejsachowicz** and **P.J. Rabier** (1991): *orientation of maps instead of spaces*. They introduce a degree for (oriented) nonlinear Fredholm maps of index zero between Banach spaces.

P. Benevieri and **M. Furi** (1997): degree for nonlinear Fredholm maps of index zero between Banach manifolds with *a different notion of orientation of maps* with respect to the previous one given by F. P. and R..

P. Benevieri and **M. Furi** (2005): degree for locally compact perturbations (extended to condensing) perturbations of nonlinear Fredholm maps of index zero between Banach spaces.

P. Rabier and **M. Salter** (2005): (with a slight different approach) degree for completely continuous perturbations of nonlinear Fredholm maps of index zero between Banach spaces.

Let us sketch the costruction of the degree for the locally compact perturbations of nonlinear Fredholm maps between Banach spaces.

The problem of orientation:

we define a concept of orientation for linear Fredholm operators of index zero between Banach spaces:

E and F real Banach spaces, $L : E \to F$ linear Fredholm operator of index zero.

$E = E_1 \oplus \mathrm{Ker}\, L$, $F = L(E_1) \oplus F_2$. Thus L can be represented by a matrix of operators

$$\begin{pmatrix} L_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where L_{11} is an isomorphism.

Consider two linear operators $A, B : E \to F$, represented by

$A = \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix}$ e $B = \begin{pmatrix} 0 & 0 \\ 0 & B_1 \end{pmatrix}$, with $A_1, B_1 : \operatorname{Ker} L \to F_2$ isomorphisms.

We say that A and B are *equivalent* if the determinant of $B_1^{-1} A_1$ is positive (it does not depend on the choice of the basis on $\operatorname{Ker} L$).

The set of these "correctors" (as A and B above) has two equivalence classes. A choice of one of them is an *orientation* of L.

Let $g : E \to F$ be a (nonlinear) Fredholm map of index zero. Assume $Dg(x)$ oriented for any $x \in E$.

By a notion of "continuous transport" of the orientation $Dg(x)$, moving x in E, we define an *orientation* of g as a "continuous" choice of an orientation of $Dg(x)$ for any $x \in E$.

If, for a given x, $Dg(x)$ is an isomorphism, we assign

$$\begin{cases} \text{sign}\, Df(x) = 1 & \text{if the trivial operator belongs to the chosen} \\ & \text{orientation of } Dg(x) \\ \text{sign}\, Df(x) = -1 & \text{otherwise.} \end{cases}$$

If U is open in E, $g^{-1}(y) \cap U$ is compact and y is a regular value for g in U, we define

$$\text{(5)} \qquad \deg(g, U, y) = \sum_{x \in g^{-1}(y) \cap U} \text{sign}\, Dg(x).$$

The degree is then extended to the triples (g, U, y) with y not necessarily regular value of g in U.

Degree for locally compact perturbations of nonlinear Fredholm maps of index zero between Banach spaces.

We call these maps *quasi-Fredholm maps*.

We define a concept of orientation for the quasi-Fredholm maps.

Definition 1. Let $g : \Omega \to F$ be a Fredholm map of index zero and $k : E \to F$ a locally compact map. The map $f : E \to F$, defined by $f = g - k$, is called a *quasi-Fredholm map* and g is a *smoothing map* of f.

The following definition provides an extension to quasi-Fredholm maps of the concept of orientation of Fredholm maps.

Definition 2. An *orientation* for a quasi-Fredholm map $f = g - k : E \to F$ is an orientation of the smoothing map g.

Let $f = g - k$ be an oriented quasi-Fredholm map, (f, U, y) an admissible triple (i.e. $f^{-1}(y) \cap U$ compact).

Step 1. (*Finite-dimensional reduction process*) Suppose $k(U)$ contained in a finite dimensional subspace of F.

Let Z be a finite-dimensional subspace of F and W an open neighborhood of $f^{-1}(y)$ in U, with g transverse to Z in W.

Assume that Z contains y and $k(U)$ and suppose Z oriented.

$M := g^{-1}(Z) \cap W$ is a C^1 manifold and $\dim M = \dim Z$.

M can be oriented with an orientation induced by the orientations of g and Z. Then we define

(6) $$\deg_{qF}(f, U, y) = \deg_B(f|_M, M, y),$$

The above definition is well posed in the sense that the right hand side of (**??**) is independent of the choice of the smoothing map g, the open set W and the subspace Z.

Step 2. The degree is extended to general admissible triples.

Fundamental properties

i) (*Normalization*) Let $L : E \to F$ be a naturally oriented isomorphism. Then

$$\deg_{qF}(L, E, 0) = 1 \quad (= \operatorname{sign} L).$$

ii) (*Additivity*) Given an admissible triple (f, U, y) and two disjoint open subsets U_1, U_2 of U such that $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$, then,

$$\deg_{qF}(f, U, y) = \deg_{qF}(f|_{U_1}, U_1, y) + \deg_{qF}(f|_{U_2}, U_2, y).$$

iii) (*Homotopy invariance*) Let $H : U \times [0,1] \to F$ be an oriented quasi-Fredholm homotopy. Let $y : [0,1] \to F$ be a continuous path. If the set

$$\{(x,t) \in U \times [0,1] : H(x,t) = y(t)\}$$

is compact, then $\deg_{qF}(H_t, U, y(t))$ does not depend on $t \in [0,1]$.

Main result: there exists at most one real map defined in the class of quasi-Fredholm admissible triples which verifies the three fundamental properties: normalization, additivity and homotopy invariance. Thus, such a map turns out to be integer valued and necessarily coincides with the degree for oriented quasi-Fredholm maps

Sketch of the process.

We denote by $\mathcal{T}$ the family of all admissible triples and by $d : \mathcal{T} \to \mathbb{R}$ a map verifying the three fundamental properties.

Step 1. We prove that, if L is an oriented isomorphism, then

$$d(L, E, 0) = \operatorname{sign} L. \tag{7}$$

Step 2. Using the above equality, the additivity and the homotopy invariance property, we show that, for every triple (f, U, y) such that $f|_U$ is C^1 and y is a regular value of f in U, we have

$$d(f, U, y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sign} Df(x). \tag{8}$$

Step 3. Then we prove the uniqueness of d on the subfamily $\mathcal{S}$ of $\mathcal{T}$ of the triples (f, U, y) such that f is C^1 on U.

Step 4. We show that the uniqueness of d on $\mathcal{S}$ implies the uniqueness of d on the subfamily $\mathcal{T}$ of those admissible triples (f, U, y) such that we can write $f = g - k$ with $k(U)$ is contained in a finite dimensional subspace of F.

(In this step it is contained one of the most important difficulties of process.)

As it is well know, a real continuous map, defined in a compact subset of $\mathbb{R}^n$, can be approximated, in the supremum norm, by a smooth map, defined on the whole $\mathbb{R}^n$. As far as we know, an analogous result does not hold if $\mathbb{R}^n$ is replaced by a general Banach space E, unless the compact domain of the map is contained in a finite-dimensional subspace of E (recall that any finite-dimensional subspace of E is the image of a bounded linear projector).

Thanks to the following lemma by Pejsachowicz and Rabier

(in *A substitute for the Sard-Smale theorem in the* C^1 *case*, J. Anal. Math. 76 (1998), 265–288),

an approximation result holds true even in the case when the domain of the map *is contained in a finite-dimensional submanifold of* E.

Lemma 3 (Pejsachowicz–Rabier)**.** *Consider a finite-dimensional* C^1 *submanifold* M *of* E *and a compact subset* K *of* M*. Then, there exist a finite-dimensional subspace* E_1 *of* E *and a* C^1 *diffeomorphism* $w : E \to E$ *such that* $w(K) \subseteq E_1$*.*

As a consequence we obtain the following proposition.

Proposition 4. *Let K be a compact subset of E. Assume that there exists a finite dimensional submanifold M of E containing K. Let $\phi : K \to \mathbb{R}$ be a continuous map. Given a positive ε, there exists a bounded C^1 map $\eta : E \to \mathbb{R}$ such that*

$$\sup_{x \in K} |\phi(x) - \eta(x)| < \varepsilon.$$

Let now $f = g - k$ be an oriented quasi-Fredholm map and (f, U, y) an admissible triple (i.e. $f^{-1}(y) \cap U$ compact).

Suppose $k(U)$ contained in a finite dimensional subspace of F.

Let Z be a finite-dimensional subspace of F and W an open neighborhood of $f^{-1}(y)$ in U, with g transverse to Z in W.

Assume that Z contains y and $k(U)$ and suppose Z oriented.

$M := g^{-1}(Z) \cap W$ is a C^1 manifold and $\dim M = \dim Z$.

We construct a suitable compact subset S of E, with $f^{-1}(y) \cap U \subseteq S \subseteq M$, and a compact C^1 map $h : E \to F$, having image contained in Z, which is an approximation of k on S.

Applying the homotopy invariance property we have

$$d(g - k, U, y) = d(g - h, U, y). \tag{9}$$

Step 7. In this final step we conclude the process, showing the uniqueness of d on the whole family $\mathcal{T}$.

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Finally, since the function deg verifies the three fundamental properties, one will have $d = \text{deg}$.

Thank you!