Gagliardo-Nirenberg estimates for localizable Hardy-Sobolev spaces

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Question: if a function f belongs to a Sobolev space $W^{1,p}(\mathbb{R}^N)$, does f automatically belong to certain other Sobolev space?

In other words, what is the validity of the estimate

$$\|f\|_{L^q(\mathbb{R}^N)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^N)}, \quad \forall\, f \in C_c^\infty(\mathbb{R}^N), \tag{1}$$

where C is independent of q and N ?

If the estimate (1) holds, the relation between p and q is given by $\frac{1}{q} = \frac{1}{p} - \frac{1}{N}$ that implies $p < N$ and $p < q$.

Fix $1 \leq p < N$. We define the Sobolev conjugate of p by

$$p^* \doteq \frac{Np}{N-p} \Longleftrightarrow \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Gagliardo-Nirenberg estimate

Assume $1 \leq p < N$. There exists a constant C, depending only on p and N, such that

$$\|f\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^N)}, \quad \forall\, f \in C_c^\infty(\mathbb{R}^N).$$

In this lecture we are interested in the following question: what happens if we consider $0 < p < 1$?

Bad idea because $L^p(\mathbb{R}^N)$ is not normed with functional $\|f\|_{L^p(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |f(x)|^p dx)^{\frac{1}{p}}$. Solution: change the space!

For any $\Phi \in \mathcal{S}(\mathbb{R}^N)$ and tempered distribution f, we define the maximal function $M_\Phi f$ by

$$M_\Phi f(x) = \sup_{t>0} |(f * \Phi_t)(x)|,$$

where $\Phi_t(x) = t^{-N}\Phi(x/t)$.

Hardy-Spaces (Fefferman-Stein, 1972)

Let $0 < p \leq \infty$. A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^N)$ belongs to $H^p(\mathbb{R}^N)$. $\equiv$. there is $\Phi \in S$ with $\int_{\mathbb{R}^N} \Phi(x)dx \neq 0$ such that

$$\|f\|_{H^p(\mathbb{R}^N)} \doteq \|M_\Phi f\|_{L^p(\mathbb{R}^N)} < \infty$$

Moreover $\|f\|_{H^p(\mathbb{R}^N)}$ is

- independent of the choice of Φ.
- a quasi-norm when $0 < p < 1$.

General properties of H^p spaces

(i) $f \in H^p(\mathbb{R}^N)$ then $M_\Phi f \in L^p(\mathbb{R}^N)$ $\forall$ $\Phi \in \mathcal{S}$ with $\int_{\mathbb{R}^N} \Phi \neq 0$.

(ii) H^p is complete in the metric $d(f,g) = \|f-g\|_{H^p}^p$, $p \leq 1$.

Natural extension

(iii) $H^p(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ for $1 < p \leq \infty$.

(iv) $H^1(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$ strictly.

(v) Singular integral operators are bounded from H^p to itself.

Special property

(vi) $f \in L^1(\mathbb{R}^N) \cap H^p(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} x^\alpha f(x)dx = 0,$$

whenever $|\alpha| \leq N(p^{-1} - 1)$.

Theorem

Suppose that $f \in H^p(\mathbb{R}^N)$ with $\nabla f \in H^p(\mathbb{R}^N)$ for $0 < p < N$. Then $f \in H^{p^*}(\mathbb{R}^N)$ with

$$\|f\|_{H^{p^*}(\mathbb{R}^N)} \leq C_{p,q} \|\nabla f\|_{H^p(\mathbb{R}^N)}. \tag{2}$$

Basically the proof is consequence of the next result when $\alpha = 1$.

Proposition [Krantz, 1982]

Let $0 < \alpha < N$ and the Riesz potential I_α defined by $(I_\alpha)\hat{}(x) = |x|^{-\alpha}\hat{f}(x)$. If $0 < p < N/\alpha$ then $I_\alpha f \in H^q(\mathbb{R}^N)$ with

$$\|I_\alpha f\|_{H^q(\mathbb{R}^N)} \leq C_{p,q} \|f\|_{H^p(\mathbb{R}^N)}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}. \tag{3}$$

Applications and Hardy spaces

In a high number of applications in PDE's it is necessary localize elements in $H^p(\mathbb{R}^N)$.

Remark

If $\eta \in C_c^\infty(\mathbb{R}^N)$ and $f \in H^p(\mathbb{R}^N)$ then $\eta.f$ is not necessarily in $H^p(\mathbb{R}^N)$ because the (global) moment conditions may be violated.

A way to get around this problem it is enough define the truncated maximal function

$$m_\Phi f(x) = \sup_{0<t<1} |(f * \Phi_t)(x)|,$$

where $\Phi_t(x) = t^{-N}\Phi(x/t)$.

Localizable Hardy spaces (Goldberg, 1979)

Let $0 < p \leq \infty$. A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^N)$ belongs to $h^p(\mathbb{R}^N)$. $\equiv$. there is $\Phi \in S$ with $\int_{\mathbb{R}^N} \Phi(x)dx \neq 0$ such that

$$\|f\|_{h^p(\mathbb{R}^N)} \doteq \|m_\Phi f\|_{L^p(\mathbb{R}^N)} < \infty$$

General properties:

(i) $f \in h^p(\mathbb{R}^N)$ then $m_\Phi f \in L^p(\mathbb{R}^N)$ $\forall$ $\Phi \in \mathcal{S}$ with $\int_{\mathbb{R}^N} \Phi \neq 0$.

(ii) h^p is complete in the metric $d(f,g) = \|f-g\|_{h^p}^p$, $p \leq 1$.

(iii) $H^p(\mathbb{R}^N) = h^p(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ for $1 < p \leq \infty$.

(iv) $H^p(\mathbb{R}^N) \subset h^p(\mathbb{R}^N)$ strictly for $0 < p \leq 1$.

(v) $h^1(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$ strictly.

(vi) Pseudo-differential operators with symbol in Hormander class $S^0_{1,\delta}(\mathbb{R}^N)$ for $0 \leq \delta < 1$ are bounded on $h^p(\mathbb{R}^N)$.

Main facts

(I) $\eta \in C_c^\infty(\mathbb{R}^N)$ and $f \in h^p(\mathbb{R}^N)$ then $\eta.f \in h^p(\mathbb{R}^N)$.

(II) Let $\Psi \in \mathcal{S}(\mathbb{R}^N)$ and suppose that $\int \Psi = 1$ and $\int x^\alpha \Psi(x)dx = 0$ for all $\alpha \neq 0$. If $f \in h^p(\mathbb{R}^N)$ then

$$f = (f - \Psi * f) + \Psi * f,$$

such that $\|f - \Psi * f\|_{H^p(\mathbb{R}^N)} \leq \|f\|_{h^p(\mathbb{R}^N)}$. In particular

$$f \in h^p(\mathbb{R}^N) \Rightarrow f - \Psi * f \in H^p(\mathbb{R}^N).$$

(III) $f \in h^p(\mathbb{R}^N)$ has compact support then $f \in H^p$ modulo a $C_c^\infty(\mathbb{R}^N)$ function.

Let $0 < p \leq \infty$ and consider on $C_c^\infty(\mathbb{R}^N)$ the functional

$$\|f\|_{h^{1,p}(\mathbb{R}^N)} \doteq \sum_{j=1}^{N} \|\partial_j f\|_{h^p(\mathbb{R}^N)} = \|\nabla f\|_{h^p(\mathbb{R}^N)}.$$

Localizable Hardy-Sobolev space

$h^{1,p}(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ for the norm $\|f\|_{h^{1,p}(\mathbb{R}^N)}$ identified with a subspace of $\mathcal{S}'(\mathbb{R}^N)$.

If $\mathfrak{h}^{1,\mathfrak{p}}(\mathbb{R}^N)$ is defined as tempered distributions $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\nabla f \in h^p(\mathbb{R}^N)$ equipped with the seminorm $\|\nabla f\|_{h^p}$ then

$$h^{1,p}(\mathbb{R}^N) \simeq \mathfrak{h}^{1,\mathfrak{p}}(\mathbb{R}^N) / \{const.\}$$

In the literature the *homogeneous* space $h^{1,p}(\mathbb{R}^N)$ is often denoted by $\dot{h}^{1,p}(\mathbb{R}^N)$.

Remarks:

(1) If $\frac{N}{N+1} \leq p \leq 1$ then $h^{1,p}(\mathbb{R}^N) \subset L^1_{loc}(\mathbb{R}^N)$.

(2) There exist $C > 0$ such that the non-homogeneous estimate

$$\|f\|_{h^{p^*}(\mathbb{R}^N)} \leq C\left(\|f\|_{h^p(\mathbb{R}^N)} + \|\nabla f\|_{h^p(\mathbb{R}^N)}\right)$$

holds.

Question: Is Gagliardo-Nirenberg valid for $h^p(\mathbb{R}^N)$?
The estimate

$$\|f\|_{h^{p^*}(\mathbb{R}^N)} \leq C\|\nabla f\|_{h^p(\mathbb{R}^N)}$$

fails because of growth at infinity.

Counter example (Krantz, 1982)

The Riesz potential I_1 is not bounded from $h^p(\mathbb{R}^N)$ to $h^{p^*}(\mathbb{R}^N)$.

Therefore, if we consider distributions supported in balls then the **obstacle vanishes**! Let $\mathcal{E}'(B)$ the space of distributions with compact support contained in B generic ball and consider the standard notation

$$h^p_c(B) \doteq h^p(\mathbb{R}^N) \cap \mathcal{E}'(B) \;\text{ and }\; h^{1,p}_c(B) \doteq h^{1,p}(\mathbb{R}^N) \cap \mathcal{E}'(B).$$

Suppose $\frac{N}{N+1} < p \leq 1$ that implies

(1) $p^* > 1$ and so $h^{p^*}(\mathbb{R}^N) = L^{p^*}(\mathbb{R}^N)$.
(2) $h^{1,p}(\mathbb{R}^N) \subset L^1_{loc}(\mathbb{R}^N)$.

In this case, the embedding on Lizorkin-Triebel spaces $F^s_{p,q}$ and characterizations of the localizable Hardy-Sobolev spaces prove that

$$h^{1,p}_c(B) \subset h^{1,p}(B) = F^1_{p,2}(B) \subset\subset F^0_{q,2}(B) = h^q(B)$$

for $p \leq q < p^*$.

Theorem [Hoepfner, Hounie and ——, 2013]

Let $N \geq 2$, $0 < p \leq 1$ and $p^* = \frac{pN}{N-p}$.

(i) If $B \subset \mathbb{R}^N$ is an open ball, there is a continuous embedding

$$h_c^{1,p}(B) \subset h_c^{p^*}(B).$$

In particular,

$$\|f\|_{h^{p^*}(\mathbb{R}^N)} \leq C(B)\|\nabla f\|_{h^p(\mathbb{R}^N)}, \quad \forall\, f \in C_c^\infty(B).$$

(ii) For $p \leq q < p^*$, the embedding

$$h_c^{1,p}(B) \subset\subset h_c^q(B)$$

is compact.

Applications

Corollary 1

Let $N \geq 2$ and $\frac{N}{N+1} < p \leq 1$. Then for each $\epsilon > 0$ exist $\rho = \rho(\epsilon) > 0$ such that

$$\|\phi\|_{h^p(\mathbb{R}^N)} \leq \epsilon \|\nabla\phi\|_{h^p(\mathbb{R}^N)}, \quad \forall\, \phi \in C_c^\infty(B(0,\rho)).$$

Now consider $L_1, \ldots, L_n$ the system of vector fields

(i) with smooth coefficients defined on a neighborhood Ω of the origin $0 \in \mathbb{R}^N$;

(ii) linearly independent.

Definition

The system $\{L_1, \ldots, L_n\}$ is *elliptic* if for any real 1-form ω such that $\langle \omega, L_j \rangle = 0 \Rightarrow \omega = 0$.

Corollary 2

Let $0 < p < N$. If the system of vector fields $L_1, ..., L_n$, $n \geq 2$, is elliptic then every point $x_0 \in \Omega$ is contained in an open neighborhood $\mathcal{U} \subset \Omega$ such that for some $C > 0$

$$\|u\|_{h^{p^*}(\mathbb{R}^N)} \leq C \left(\sum_{j=1}^{n} \|L_j u\|_{h^p(\mathbb{R}^N)} \right), \quad \forall\, u \in C_c^\infty(\mathcal{U}), \tag{4}$$

holds.

Remark: Hounie and Picon (2011) proved the converse of the last result when $1 \leq p < N$: if the estimate (4) holds then the system must be elliptic on $\mathcal{U}$.

References

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FIM!