ON THE APPROXIMATE VARIATIONAL MEASURE

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SOME DEFINITIONS

By a tagged interval we mean a pair (I, x), where $x \in I \subset \mathbb{R}$.

By a division in an interval $[a, b]$ we mean a finite collection $\{(I_i, x_i)\}_{i=1}^{k}$ of tagged intervals, where intervals I_i therein are pairwise nonoverlapping.

A division in $[a, b]$ is called a partition of $[a, b]$ if $\bigcup_{i=1}^{k} I_i = [a, b]$.

Having a function $\delta\colon \mathbb{R} \to (0, \infty)$, called a gauge, we say that a division $\{(I_i, x_i)\}_{i=1}^{k}$ is δ-fine if for each i,

$$I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i)).$$

We say a division $\{(I_i, x_i)\}_{i=1}^{k}$ is anchored in a set $E \subset \mathbb{R}$ if $x_i \in E$ for each i.

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KURZWEIL–HENSTOCK INTEGRAL

DEFINITION. ***Kurzweil 1957 & Henstock 1960***

We call a function $f\colon [a,b] \to \mathbb{R}$*, H-integrable, with the integral* $\mathbf{I} = (H)\int_a^b f \in \mathbb{R}$*, if for each* $\varepsilon > 0$ *there exists a gauge* δ*, such that for every* δ*-fine partition* $\{(I_i, x_i)\}_{i=1}^k$ *of* $[a,b]$*,*

$$\left| \sum_{i=1}^{k} f(x_i)|I_i| - \mathbf{I} \right| < \varepsilon.$$

Set the indefinite integral of f:

$$F(x) = (H)\int_a^x f, \qquad x \in [a,b].$$

How to characterize F*?*

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VARIATIONAL MEASURE

Let $F\colon [a,b] \to \mathbb{R}$.

By $|E|_F$ we mean the variational measure of $E \subset [a,b]$ induced by F; i.e.,

$$|E|_F = \inf_{\delta} \sup_{\mathcal{P}} \sum_{i=1}^{k} |\Delta F(I_i)|,$$

where sup runs over all δ-fine divisions $\{(I_i, x_i)\}_{i=1}^{k}$ anchored in E.

The function F is said to be *SL* (after *Strong Lusin Condition*) if $|\cdot|_F$ is absolutely continuous; i.e.,

$$|E|_F = 0 \quad \text{for every (Lebesgue) nullset} \quad E \subset [a,b].$$

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A CHARACTERIZATION OF H-INTEGRAL

THEOREM. ***Ene 1994, Bongiorno & Di Piazza & Skvortsov 1995.***
Let $F\colon [a,b] \to \mathbb{R}$, $F(a) = 0$. *TAE:*

❶ F *is an indefinite Kurzweil–Henstock integral (of* F'*),*

❷ F *is SL.*

Key steps of the proof:

- notice that $|\cdot|_F$ is absolutely continuous on each set

$$\{x \in [a,b] : F'(x) \text{ exists and } F'(x) \le n\}, \quad n \in \mathbb{N};$$

so it's enough to prove

- $|\cdot|_F$ is absolutely continuous $\Longrightarrow$ F is almost everywhere differentiable.

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LEMMA. ***Bongiorno & Di Piazza & Skvortsov 1995.***
$|\cdot|_F$ is absolutely continuous $\Longrightarrow$ F is VBG_*, so almost everywhere differentiable.

A function $F\colon [a,b] \to \mathbb{R}$ is said to be VBG_* if $[a,b] = \bigcup_{n=1}^{\infty} E_n$, where, for each n,

$$\sum_{i=1}^{k} \omega_F(I_i) < M_n$$

for any collection $\{I_i\}_{i=1}^{k}$ of nonoverlapping intervals with both endpoints in E_n.

LEMMA. ***Lusin ?***
Every VBG_*-function is almost everywhere differentiable.

FACT. An $F\colon [a,b] \to \mathbb{R}$ is VBG_*-function iff $|\cdot|_F$ is σ-finite on a co-countable subset of $[a,b]$.

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APPROXIMATE KURZWEIL–HENSTOCK INTEGRAL

Let $S \subset \mathbb{R}$. We say a tagged interval $([y, z], x)$ is S-fine if $y, z \in S$.

Let $\mathcal{C} = \{S_x : x \in \mathbb{R}\}$ be a collection of sets. We say a division $\{(I_i, x_i)\}_{i=1}^k$ is $\mathcal{C}$-fine if for each i, (I_i, x_i) is S_{x_i}-fine.

Let a set $A \subset \mathbb{R}$ be measurable, $x \in \mathbb{R}$. The density of A at x is, if it exists, the value

$$d(A, x) = \lim_{h \to 0} \frac{|A \cap [x-h, x+h]|}{2h}.$$

Let $\Delta(x)$, $x \in \mathbb{R}$, be the collection of measurable sets $E \subset \mathbb{R}$ with $d(E, x) = 1$.

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For any collection $\mathcal{C} = \{S_x \in \Delta(x) : x \in \mathbb{R}\}$, there is a $\mathcal{C}$-fine partition of any $[a, b] \subset \mathbb{R}$.

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We call a function $f\colon [a, b] \to \mathbb{R}$, AH-integrable, with the integral $\mathbf{I} = (AH)\int_a^b f \in \mathbb{R}$, if for each $\varepsilon > 0$ there exists a collection $\mathcal{C} = \{S_x \in \Delta(x) : x \in \mathbb{R}\}$, such that for every $\mathcal{C}$-fine partition $\{(I_i, x_i)\}_{i=1}^k$ of $[a, b]$,

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APPROXIMATE VARIATIONAL MEASURE AND *VBG*

To this aim, one proves

- F is ASL implies F is measurable;
- F is $ASL \Longrightarrow F$ has VBG property on each nullset;
- **LEMMA.** ***Ene 1998***
  A measurable F is VBG iff it is VBG on every Lebesgue nullset $E \subset [a, b]$.
- **LEMMA.** ***Denjoy–Khintchine 1916***
  A measurable VBG function is a.e. approximately differentiable.

A function $F\colon [a, b] \to \mathbb{R}$ is said to be VBG if $[a, b] = \bigcup_{n=1}^{\infty} E_n$, where, for each n,

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Take $P = [a,b] \setminus \bigcup_{n=1}^{\infty} E_n$, $|P| = 0$, $E_n = \operatorname{cl} E_n$, $F \restriction E_n$ – continuous.

$F \notin VBG[a,b]$ implies $F \notin VBG(E_n)$ for some n.

D — the set of all $x \in E_n$ such that F is VBG on no $I \cap E_n$, $I \ni x$ an open interval.

D' — co-countable subset of D without unilaterally isolated points

Pick $I_1^{(1)}, \ldots, I_{m_1}^{(1)}$, $m_1 \geq 2$, with endpoints in D', so that

$$\sum_{i=1}^{m_1} \left|\Delta F\left(I_i^{(1)}\right)\right| \geq 1.$$

In $D' \cap \bigcup_{i=1}^{m_1} I_i^{(1)}$ pick intervals $I_1^{(2)}, \ldots, I_{m_2}^{(2)}$ with endpoints in D', so that

① each of $I_i^{(1)}$, $i = 1, \ldots, m_1$, contains at least two of $I_j^{(2)}$, $j = 1, \ldots, m_2$,

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And so on. With a category argument one can show that on the nullset

$$N = D \cap \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{m_k} I_i^{(k)}$$

F is not VBG. ∎

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$[0,1] = \{p_\alpha\}_{\alpha<\Omega}$

$\{G_\alpha\}_{\alpha<\Omega}$ — all $\mathcal{G}_\delta$ null subsets of $[0,1]$

Put $H_0 = G_0$ and for each $\alpha < \Omega$ take an $\tilde{\alpha} < \Omega$ such that

$$G_{\tilde{\alpha}} \supset \bigcup_{\beta<\alpha} H_\beta \cup G_\alpha$$

and

$$G_{\tilde{\alpha}} \setminus \bigcup_{\beta<\alpha} H_\beta \quad \text{is uncountable.} \tag{1}$$

Define $H_\alpha = G_{\tilde{\alpha}}$.

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$\{G_\alpha\}_{\alpha<\Omega}$ — all $\mathcal{G}_\delta$ null subsets of $[0,1]$

Put $H_0 = G_0$ and for each $\alpha < \Omega$ take an $\tilde{\alpha} < \Omega$ such that

$$G_{\tilde{\alpha}} \supset \bigcup_{\beta<\alpha} H_\beta \cup G_\alpha$$

and

$$G_{\tilde{\alpha}} \setminus \bigcup_{\beta<\alpha} H_\beta \quad \text{is uncountable.} \tag{2}$$

Define $H_\alpha = G_{\tilde{\alpha}}$.

$\{H_\alpha\}_{\alpha<\Omega}$ is ascending, $\quad \bigcup_{\alpha<\Omega} H_\alpha = \bigcup_{\alpha<\Omega} G_\alpha = [0,1].$

Put

$$F(x) = p_\alpha \quad \text{at} \quad x \in H_\alpha \setminus \bigcup_{\beta<\alpha} H_\beta.$$

F is VBG on each nullset $D \subset [0,1]$:

$D \subset G_\alpha \subset H_\alpha$ for some $\alpha < \Omega$. Thus $F(D) \subset \{p_\beta\}_{\beta \leqslant \alpha}$.

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Suppose $[0,1] = \bigcup_{n=1}^{\infty} E_n$ and F is VB on each E_n.

$I_n = \{p_\alpha : E_n \cap F^{-1}(p_\alpha) \text{ is infinite}\}$, $n \in \mathbb{N}$.

$|I_n| > 0$ for some n, since $[0,1] = \bigcup_{n=1}^{\infty} I_n$

Consider the indicatrix function I of $F \restriction E_n$. By the Banach indicatrix theorem,

$$\int_{-\infty}^{\infty} I \leqslant V(F \restriction E_n) < \infty.$$

On the other hand, $I(y) = \infty$ for each $y \in I_n$, whence

$$\int_{-\infty}^{\infty} I \geqslant \infty \cdot |I_n| = \infty,$$

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SOME GENERALIZATIONS RELATED TO MEASURABILITY

Extra assumptions on $\{\Delta(x)\}_{x\in\mathbb{R}}$:

❶ if $S \in \Delta(x)$ then there exists a measurable set $R \subset S$ such that

$$\underline{d}(R, x) > 0;$$

❷ if $S \in \Delta(x)$ and $d(R, x) = 1$, $R \ni x$, then

$$S \cap R \in \Delta(x).$$

THEOREM. ***VAS & PS 2012***
Let $F\colon [a,b] \to \mathbb{R}$. Assume $|\cdot|_F^{\Delta}$ is σ-finite on each nullset. Then $F\colon \mathbb{R} \to \mathbb{R}$ is measurable.

THEOREM. ***VAS & PS 2002***
If $F\colon \mathbb{R} \to \mathbb{R}$ is measurable, then $|\cdot|_F^{\mathrm{ap}}$ is σ-finite on each nullset implies it is σ-finite on $[a,b]$. In turn this implies F is almost everywhere approximately differentiable.

COROLLARY. ***VAS & PS 2012***
Assume $\mathrm{V}_F^{\mathrm{ap}}$ is σ-finite on each nullset. Then $\mathrm{V}_F^{\Delta_{\mathrm{ap}}}$ is σ-finite and F is approximately differentiable a.e.

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