Linear subsets of nonlinear sets in topological vector spaces

Daniel Pellegrino

Universidade Federal da Paraiba, Brazil

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The main goal of this talk..... ....is to provide an overview (and a propaganda!!) of a relatively new area of research connecting analysis and algebra which we call lineability....

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  enjoys this property.

- In the literature, this example is known as **Weierstrass' monster** although earlier **Bolzano** (1822) and **Cellérier** (1860) already found functions of this type!

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- **1999**: Fonf, Gurariy, and Kadeč proved that the above space can be chosen to be closed in $\mathcal{C}[0,1]$.
- Rodríguez-Piazza (**1995**), Hencl (**2000**), Bayart and Quarta (**2007**), among others, have improved these spaces by adding *extra pathologies* to Weierstrass' monster.

Lineability and Spaceability. The basics

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As we will comment later, it is not difficult to see that there may not exist a maximal λ satisfying the above definition.

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Theorem (Levine and Milman, 1940)

The subset of $\mathcal{C}[0,1]$ of all functions of bounded variation is not spaceable.

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Proc. Amer. Math. Soc. **133** *(2004) 795–803.*

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Around that time and since then, many authors have shown their interest in this topic...

where do we found lineability?

Lineability and spaceability can be investigated in several different contexts.....for instance....

Different directions in the study of lineability

Sets of zeroes of polynomials in Banach spaces

- Aron, Rueda (1997).
- Plichko, Zagorodnyuk (1998).
- Aron, Gonzalo, Zagorodnyuk (2000).
- Aron, García, Maestre (2001).
- Aron, Boyd, Ryan, Zalduendo (2003).
- Aron, Hajék (2006).
- Lourenço, Tocha (2007).

Different directions in the study of lineability

Hypercyclicity and close subjects

- Godefroy, Shapiro (1991).
- Montes (1996).
- Aron, García, Maestre (2001).
- Aron, Bès, León, Peris (2005).
- Seoane-Sepúlveda (2007).
- Aron, Conejero, Peris, Seoane-Sepúlveda (2007).
- Bernal-González (2009).
- Shkarin (2010)
- Bertoloto, Botelho, Fávaro, Jatobá (2013).

Different directions in the study of lineability

Continuous nowhere differentiable functions in $\mathcal{C}[0,1]$

- Rodríguez-Piazza (1995).
- Fonf, Gurariy, Kadeč (1999).
- Aron, García, Maestre (2001).
- Bayart, Quarta (2007).
- Bernal-González (2008).
- Aron, García, Pérez, Seoane-Sepúlveda (2009).

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Norm-attaining functionals

- Aron, García, Maestre (2001).
- Bandyopadhyay, Godefroy (2006).
- Acosta, Aizpuru, Aron, García (2007).
- –, Teixeira (2009).
- García, Puglisi (2011).

Different directions in the study of lineability

Subsets of $\mathbb{R}^{\mathbb{R}}$

- Aron, Gurariy, Seoane-Sepúlveda (2004).
- Gurariy, Quarta (2004).
- Bayart, Quarta (2007).
- Aron, Seoane-Sepúlveda (2007).
- Aron, Gorkin (2007).
- García, Palmberg, Seoane-Sepúlveda (2007).
- Aizpuru, Pérez, García, Seoane-Sepúlveda (2008).
- Azagra, Muñoz, Sánchez, Seoane-Sepúlveda (2009).
- Aron, García, Pérez, Seoane-Sepúlveda (2009).
- García, Grecu, Maestre, Seoane-Sepúlveda (2010).
- Gámez, Muñoz, Sánchez, Seoane-Sepúlveda (2010).
- Gámez, Muñoz, Seoane-Sepúlveda (2010, 2011).
- Bartoszewicz, Głąb, –, Seoane-Sepúlveda (2012).
- Enflo, Gurariy, Seoane-Sepúlveda (2012).
- Jimenez-Rodríguez, Muñoz, Seoane-Sepúlveda (2012).
- Conejero, Jimenez-Rodríguez, Muñoz, Seoane-Sepúlveda (2012).
- Albuquerque (2013).

Different directions in the study of lineability

Series and summability

- Aizpuru, Pérez, Seoane-Sepúlveda (2005).
- Aron, Pérez, Seoane-Sepúlveda (2006).
- Muñoz, Palmberg, Puglisi, Seoane-Sepúlveda (2008).
- Botelho, Diniz, Fávaro, – (2011).

Different directions in the study of lineability

Complex analysis and holomorphy

- Bernal-González (2008).
- López-Salazar (2011).
- Alves (2013).

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Measurable and non-measurable functions

- García, Seoane-Sepúlveda (2006).
- Muñoz, Palmberg, Puglisi, Seoane-Sepúlveda (2008).
- Bernal-González, Ordóñez-Cabrera (2012).
- Botelho, Fávaro, –, Seoane-Sepúlveda (2012).
- Botelho, Cariello, Fávaro, –, Seoane-Sepúlveda (2012).

Different directions in the study of lineability

Non-absolutely summing operators

- Puglisi, Seoane-Sepúlveda (2008).
- Botelho, Diniz, – (2009).
- Kitson, Timoney (2010).

Some other results and open problems....

....in view of the amount of recent works in this line, our choice of material for this talk will be merely illustrative and far from being exhaustive....

More exotic differentiable functions

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The set of differentiable nowhere monotone functions on $\mathbb{R}$ is $\aleph_0$-lineable.

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Theorem (Gámez, Muñoz, Sánchez, Seoane-Sepúlveda, 2010)

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Annulling functions in $\mathcal{C}[0,1]$ and spaceability

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Definition

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Theorem (Enflo, Gurariy, Seoane-Sepúlveda, 2012)

Let X be any infinite dimensional closed subspace of $\mathcal{C}[0,1]$. There exists:

- *An infinite dimensional closed subspace V of X, and*
- *a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0,1]$ (of pairwise different elements),*

such that $f(t_k) = 0$ for every $k \in \mathbb{N}$ and every $f \in V$.

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- There exists such subspace (with dimension of the continuum).

- Such subspace is never closed.

Sierpiński-Zygmund functions and lineability

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Theorem (Blumberg, 1922)

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. There exists a dense subset $S \subset \mathbb{R}$ such that the function $f|_S$ is continuous.

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A careful reading of the proof of this result shows that the above set S is countable.

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Naturally, we could wonder whether we can choose the subset S in Blumberg's theorem to be **uncountable**.

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Theorem (Sierpiński, Zygmund, 1923)

There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for any set $Z \subset \mathbb{R}$ of cardinality the continuum, the restriction $f|_Z$ is not a Borel map (in particular, is not continuous).

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....for every $f : \mathbb{R} \to \mathbb{R}$ there exists an uncountable set $Z \subset \mathbb{R}$ so that the restriction $f|_Z$ is continuous.

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....for every $f : \mathbb{R} \to \mathbb{R}$ there exists an uncountable set $Z \subset \mathbb{R}$ so that the restriction $f|_Z$ is continuous.

PROBLEM:... what about the lineability of the set of Sierpiński-Zygmund functions?...

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Theorem (Gámez-Merino, Munoz, Seoane-Sepúlveda, 2010)

The set of Sierpiński-Zygmund functions is $\mathfrak{c}^+$ lineable.

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Theorem (Gámez-Merino, Seoane-Sepúlveda, 2013)

The $2^{\mathfrak{c}}$-lineability of the set of Sierpiński-Zygmund functions is **undecidable**.

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Its proof uses *forcing* as the main tool.

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Theorem (Albuquerque, 2013)

The set of continuous surjective functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ is $\mathfrak{c}$-lineable.

$L_p(\Omega, \Sigma, \mu)$ spaces and lineability

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Theorem (Botelho, Favaro, –, Seoane, 2012)

$L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1]$ *is spaceable for every* $p > 0$.

$L_p(\Omega, \Sigma, \mu)$ spaces and lineability

Theorem (Botelho, Cariello, Fávaro, –, Seoane, 2012)

(Informal)...

$L_p(\Omega, \Sigma, \mu)$ spaces and lineability

Theorem (Botelho, Cariello, Fávaro, –, Seoane, 2012)

(Informal)...Let (Ω, Σ, μ) be measure space, with $\mu(\Omega) = \infty$. Under VERY natural hypothesis

$$L_p(\Omega, \Sigma, \mu) - \bigcup_{1 \leq q < p} L_q(\Omega, \Sigma, \mu)$$

is maximal spaceable.

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is maximal spaceable.

Theorem (Botelho, Cariello, Fávaro, –, Seoane, 2012)

There exists a (quite exotic) measure space (Ω, Σ, μ) with $\mu(\Omega) = \infty$ so that

$$L_p(\Omega, \Sigma, \mu) - L_q(\Omega, \Sigma, \mu)$$

with $q < p$ fails to be maximal spaceable.

Is "everything" lineable?

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EXAMPLE: A (nontrivial) set that is 1-lineable and not 2-lineable.

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If $V \subset \widehat{C}[0,1] \cup \{0\}$ is a non-trivial linear space...

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Let $\widehat{C}[0,1]$ be the subset of $C[0,1]$ of functions admitting one (and only one) absolute maximum.

If $V \subset \widehat{C}[0,1] \cup \{0\}$ is a non-trivial linear space...

...then V is 1-dimensional.....this result is due to Gurariy and Quarta.

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EXAMPLE: A non-lineable, n-lineable set ($\forall n \in \mathbb{N}$).

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The set

$$M = \bigcup_{m} \left\{ \sum_{i=2^m}^{2^{m+1}-1} a_i x^i \ : \ a_i \in \mathbb{R} \right\}$$

is n-lineable for every $n \in \mathbb{N}$ and it is not lineable in $\mathcal{C}[0,1]$.

Zeroes of polynomials and lineability

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Theorem (Plichko, Zagorodnyuk, 1998)

If X is an infinite-dimensional complex Banach space and P is an n-homogeneous polynomial on X, then $P^{-1}(0)$ contains an infinite-dimensional subspace Y.

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...what about the real case?

Zeros of polynomials and lineability

... for real scalars the situation is radically different as the polynomial $P : \ell_2 \to \mathbb{R}$ given by

$$P(x) = \sum_{j=1}^{\infty} x_j^2$$

shows.

Zeros of polynomials and lineability

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Theorem (Aron, Boyd, Ryan, Zalduendo, 2003)

Let X be a (infinite-dimensional) real Banach space which does not admit a positive definite 2-homogeneous polynomial. Then, for every 2-homogeneous polynomial $P : X \to \mathbb{R}$, there is an infinite-dimensional subspace of X on which it is identically zero.....
In other words, the zero-set of P is lineable.

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Theorem (Fernández-Unzueta, 2006)

Let X be a (infinite-dimensional) complex Banach space containing l_∞. For every n, every n-homogeneous polynomial $P : X \to \mathbb{C}$ vanishes on a non-separable subspace of X.

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Let X be a (infinite-dimensional) complex Banach space containing l_∞. For every n, every n-homogeneous polynomial $P : X \to \mathbb{C}$ vanishes on a non-separable subspace of X.

On the other hand...

Zeros of polynomials and lineability

Theorem (Aviles, Todorcevic, 2009)

There exist a Banach space X and a 2-homogeneous polynomial $P : X \to \mathbb{C}$ so that $P^{-1}(0)$ contains no nonseparable subspace.

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Theorem (Aviles, Todorcevic, 2009)

There exist a Banach space X and a 2-homogeneous polynomial $P : X \to \mathbb{C}$ so that $P^{-1}(0)$ contains no nonseparable subspace.

Theorem (Aviles, Todorcevic, 2009)

There exist a Banach space X and a 2-homogeneous polynomial $P : X \to \mathbb{C}$ so that $P^{-1}(0)$ contains both separable and nonseparable maximal subspaces.

Illustrating some arguments used in proofs of lineability and spaceability results....

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Theorem (Botelho, Favaro, –, Seoane, 2012)

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$L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1]$ is spaceable for every $p \geq 1$.

PROOF. Let us first consider the following representation of the semi-open interval $[0,1)$ as a disjoint union of intervals:

$$[0,1) = [0,1-1/2) \cup [1-1/2, 1-1/4) \cup [1-1/4, 1-1/8) \cup \cdots = \bigcup_{n=1}^{\infty} I_n,$$

where $I_n := [a_n, b_n) = \left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right)$.

Illustrating some arguments used in proofs of lineability and spaceability results....

Notice that, for every $n \in \mathbb{N}$ and every $x \in I_n$, there is a unique $x_n \in [0,1)$ such that

$$x = (1 - x_n)a_n + x_n b_n.$$

Illustrating some arguments used in proofs of lineability and spaceability results....

Notice that, for every $n \in \mathbb{N}$ and every $x \in I_n$, there is a unique $x_n \in [0,1)$ such that

$$x = (1 - x_n)a_n + x_n b_n.$$

Now, given $p > 0$, let us fix a function $f \in L_p[0,1] - \bigcup_{q>p} L_q[0,1]$, and define a sequence of functions $(f_n)_{n=1}^{\infty}$, with $f_n \colon [0,1] \longrightarrow \mathbb{R}$, as follows:

$$f_n(x) = \begin{cases} f(x_n) & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases}$$

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Also, the functions f_n are linearly independent (they have disjoint supports) and one can see that

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So $L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1]$ is $\aleph_0$-lineable.

Illustrating some arguments used in proofs of lineability and spaceability results....

To prove that $L_p[0,1] - \bigcup_{q>p} L_q[0,1]$ is spaceable the idea is to define a bounded linear and injective operator $T\colon F \longrightarrow L_p[0,1]$, where F is a Banach space, and such that

$$\overline{T(F)} \cap L_q[0,1] = \{0\}$$

for every $q > p$.

Illustrating some arguments used in the proofs of lineability....

It can be proved that

$$T\colon \ell_1 \longrightarrow L_p[0,1] \quad , \quad T((\alpha_j)_{j=1}^{\infty}) = \sum_{n=1}^{\infty} \alpha_n f_n$$

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This operator is what we need.

Final remarks

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However, and it makes everything more interesting, this technique is limited to certain situations.....

... and new techniques are needed in different situations (in some cases non-constructive approaches).

Final remarks

In our survey we have collected 260 references in some sense related to the subject.... we hope to attract more people to the field...

Thanks...