Linear subsets of nonlinear sets in topological vector spaces

Daniel Pellegrino

Universidade Federal da Paraiba, Brazil

Sao Carlos, June 2013
This talk is based on a *survey* in collaboration with
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The main goal of this talk..... ....is to provide an overview (and a propaganda!!) of a relatively new area of research connecting analysis and algebra which we call lineability....
In 1872, Weierstrass constructed a continuous nowhere differentiable function on $\mathbb{R}$.

A function such as $f(x) = \sum_{n=0}^{\infty} \cos(3^n \pi x)^{2^n}$ enjoys this property. In the literature, this example is known as Weierstrass' monster although earlier Bolzano (1822) and Cellérier (1860) already found functions of this type!
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How many examples are like Weierstrass’ monster?

Many functions enjoying this “exotic” property have been constructed since the 1800’s. Moreover:

1966: Gurariy showed that there exists an infinite dimensional linear space every nonzero function of which is continuous and nowhere differentiable on $\mathbb{R}$.

1999: Fonf, Gurariy, and Kade˘ c proved that the above space can be chosen to be closed in $C\left[0,1\right]$.

Rodríguez-Piazza (1995), Hencl (2000), Bayart and Quarta (2007), among others, have improved these spaces by adding extra pathologies to Weierstrass’ monster.
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- Rodríguez-Piazza (**1995**), Hencl (**2000**), Bayart and Quarta (**2007**), among others, have improved these spaces by adding extra pathologies to Weierstrass’ monster.
Gurariy’s results from 1966 and 1999 lead to the introduction of the following concept:

Definition (Gurariy). Let $X$ be a topological vector space and $M$ be a subset of $X$. $M$ is said to be spaceable if $M \cup \{0\}$ contains a closed infinite dimensional subspace. $M$ is said to be lineable if $M \cup \{0\}$ contains an infinite dimensional vector space. $M$ is called $\lambda$-lineable if it contains a vector space of dimension $\lambda$. As we will comment later, it is not difficult to see that there may not exist a maximal $\lambda$ satisfying the above definition.
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An old result of Levine and Milman is also illustrative...
Lineability and Spaceability. The basics

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An old result of Levine and Milman is also illustrative...

Theorem (Levine and Milman, 1940)

*The subset of \( C[0,1] \) of all functions of bounded variation is not spaceable.*
The term lineability was first used by Aron, Gurariy, Seoane-Sepulveda in


Around that time and since then, many authors have shown their interest in this topic...
where do we found lineability?

Lineability and spaceability can be investigated in several different contexts.....for instance....
Different directions in the study of lineability

Sets of zeroes of polynomials in Banach spaces

Different directions in the study of lineability

Hypercyclicity and close subjects

- Shkarin (2010)
Different directions in the study of lineability

Continuous nowhere differentiable functions in $C[0, 1]$

- Rodríguez-Piazza (1995).
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Norm-attaining functionals
- –, Teixeira (2009).
- García, Puglisi (2011).
Different directions in the study of lineability

Subsets of $\mathbb{R}^\mathbb{R}$

- Gurariy, Quarta (2004).
- Bayart, Quarta (2007).
- Conejero, Jimenez-Rodríguez, Muñoz, Seoane-Sepúlveda (2012).
Different directions in the study of lineability

Series and summability

different directions in the study of lineability

complex analysis and holomorphy

Different directions in the study of lineability

Complex analysis and holomorphy


Measurable and non-measurable functions

Different directions in the study of lineability

Non-absolutely summing operators

- Kitson, Timoney (2010).
Some other results and open problems....

....in view of the amount of recent works in this line, our choice of material for this talk will be merely illustrative and far from being exhaustive....
More exotic differentiable functions

Theorem (Aron, Gurariy, Seoane-Sepúlveda, 2004)
The set of differentiable nowhere monotone functions on $\mathbb{R}$ is $\aleph_0$-lineable.

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Annulling functions in $C[0, 1]$ and spaceability

Definition
A function $f \in C[0, 1]$ is said to be an annulling function if $f$ has infinitely many zeros in $[0, 1]$.

Theorem (Enflo, Gurariy, Seoane-Sepúlveda, 2012)
Let $X$ be any infinite dimensional closed subspace of $C[0, 1]$.
There exists:
- An infinite dimensional closed subspace $V$ of $X$, and
- a sequence $\{t_k\}_{k \in \mathbb{N}}$ (of pairwise different elements),

such that $f(t_k) = 0$ for every $k \in \mathbb{N}$ and every $f \in V$. 

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- There exists such subspace (with dimension of the continuum).
- Such subspace is never closed.
Theorem (Blumberg, 1922)

Let \( f : \mathbb{R} \to \mathbb{R} \) be an arbitrary function. There exists a dense subset \( S \subset \mathbb{R} \) such that the function \( f|_S \) is continuous.

A careful reading of the proof of this result shows that the above set \( S \) is countable. Naturally, we could wonder whether we can choose the subset \( S \) in Blumberg’s theorem to be uncountable.

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Sierpiński-Zygmund functions and lineability

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A very nice (although partial in some sense) negative answer to this was given by Sierpiński and Zygmund:
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**Theorem (Sierpiński, Zygmund, 1923)**

There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for any set $Z \subset \mathbb{R}$ of cardinality the continuum, the restriction $f|_Z$ is not a Borel map (in particular, is not continuous).
Shinoda proved in 1973 that under some axiomatic hypothesis (including the negation of the Continuum Hypothesis)....
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....for every $f : \mathbb{R} \to \mathbb{R}$ there exists an uncountable set $Z \subset \mathbb{R}$ so that the restriction $f|_Z$ is continuous.

**PROBLEM:**... what about the lineability of the set of Sierpiński-Zygmund functions?...
The set of Sierpiński-Zygmund functions is $c^+$ lineable.

A very interesting result in this direction is the following theorem...

Theorem (Gámez-Merino, Seoane-Sepúlveda, 2013)

The $2^{c^+}$-lineability of the set of Sierpiński-Zygmund functions is undecidable. Its proof uses forcing as the main tool.
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Continuous surjective functions and lineability

The following recent result is inspired in the famous Peano’s space-filling curve.

Theorem (Albuquerque, 2013)
The set of continuous surjective functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ is $c$-lineable.
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**Theorem (Albuquerque, 2013)**

*The set of continuous surjective functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ is $c$-lineable.*
$L_p(\Omega, \Sigma, \mu)$ spaces and lineability

Theorem (Botelho, Favaro, –, Seoane, 2012)

$L_p[0,1] \bigcup q > p L_q[0,1]$ is spaceable for every $p > 0$.
$L_p(\Omega, \Sigma, \mu)$ spaces and lineability

**Theorem (Botelho, Favaro, –, Seoane, 2012)**

$L_p[0, 1] \setminus \bigcup_{q > p} L_q[0, 1]$ is spaceable for every $p > 0$. 
Theorem (Botelho, Cariello, Fávaro, –, Seoane, 2012)

(Informal)...

Let $(\Omega, \Sigma, \mu)$ be measure space, with $\mu(\Omega) = \infty$. Under very natural hypothesis $L^p(\Omega, \Sigma, \mu) - \bigcup_{1 \leq q < p} L^q(\Omega, \Sigma, \mu)$ is maximal spaceable.

Theorem (Botelho, Cariello, Fávaro, –, Seoane, 2012)

There exists a (quite exotic) measure space $(\Omega, \Sigma, \mu)$ with $\mu(\Omega) = \infty$ so that $L^p(\Omega, \Sigma, \mu) - L^q(\Omega, \Sigma, \mu)$ with $q < p$ fails to be maximal spaceable.
**Theorem (Botelho, Cariello, Fávaro, –, Seoane, 2012)**

*(Informal)...Let \((\Omega, \Sigma, \mu)\) be measure space, with \(\mu(\Omega) = \infty\). Under VERY natural hypothesis*

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with \(q < p\) fails to be maximal spaceable.
Is “everything” lineable?

Example: A (nontrivial) set that is 1-lineable and not 2-lineable.

Let $\hat{C}[0,1]$ be the subset of $C[0,1]$ of functions admitting one (and only one) absolute maximum.

If $V \subset \hat{C}[0,1] \cup \{0\}$ is a non-trivial linear space...

...then $V$ is 1-dimensional......this result is due to Gurariy and Quarta.

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EXAMPLE: A non-lineable, $n$-lineable set ($\forall n \in \mathbb{N}$).
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**Example:** A non-lineable, $n$-lineable set $(\forall n \in \mathbb{N})$.

The set

$$M = \bigcup_m \left\{ \sum_{i=2^m}^{2^m+1-1} a_i x^i : a_i \in \mathbb{R} \right\}$$

is $n$-lineable for every $n \in \mathbb{N}$ and it is not lineable in $C[0,1]$. 
Zeroes of polynomials and lineability

One of the starting points of the connection between zeroes of polynomials and lineability is probably the following...

Theorem (Plichko, Zagorodnyuk, 1998)
If $X$ is an infinite-dimensional complex Banach space and $P$ is an $n$-homogeneous polynomial on $X$, then $P^{-1}(0)$ contains an infinite-dimensional subspace $Y$.

...what about the real case?

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...what about the real case?
... for real scalars the situation is radically different as the polynomial $P : \ell_2 \to \mathbb{R}$ given by

$$P(x) = \sum_{j=1}^{\infty} x_j^2$$

shows.
The following result is proved via a non-constructive approach (it uses Zorn’s Lemma)
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**Theorem (Aron, Boyd, Ryan, Zalduendo, 2003)**

Let $X$ be a (infinite-dimensional) real Banach space which does not admit a positive definite $2$-homogeneous polynomial. Then, for every $2$-homogeneous polynomial $P : X \to \mathbb{R}$, there is an infinite-dimensional subspace of $X$ on which it is identically zero.... In other words, the zero-set of $P$ is lineable.
One can ask about separability of the subspaces inside $P^{-1}(0)$....
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**Theorem (Fernández-Unzueta, 2006)**

Let $X$ be a (infinite-dimensional) complex Banach space containing $l_{\infty}$. For every $n$, every $n$-homogeneous polynomial $P : X \to \mathbb{C}$ vanishes on a non-separable subspace of $X$. 
One can ask about separability of the subspaces inside $P^{-1}(0)$. 

**Theorem (Fernández-Unzueta, 2006)**

Let $X$ be a (infinite-dimensional) complex Banach space containing $l_\infty$. For every $n$, every $n$-homogeneous polynomial $P : X \to \mathbb{C}$ vanishes on a non-separable subspace of $X$.

On the other hand...
Theorem (Aviles, Todorcevic, 2009)

There exist a Banach space $X$ and a 2-homogeneous polynomial $P : X \rightarrow \mathbb{C}$ so that $P^{-1}(0)$ contains no nonseparable subspace.
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Theorem (Aviles, Todorcevic, 2009)

There exist a Banach space $X$ and a 2-homogeneous polynomial $P : X \to \mathbb{C}$ so that $P^{-1}(0)$ contains both separable and nonseparable maximal subspaces.
Illustrating some arguments used in proofs of lineability and spaceability results....

Theorem (Botelho, Favaro, –, Seoane, 2012)

\[ L^p[0,1] \cup L^q[0,1] \text{ is spaceable for every } p \geq 1. \]

Proof. Let us first consider the following representation of the semi-open interval \([0,1)\) as a disjoint union of intervals:

\[ [0,1) = [0,1-1/2) \cup [1-1/2,1-1/4) \cup [1-1/4,1-1/8) \cup \cdots = \bigcup_{n=1}^{\infty} I_n, \]

where \(I_n := [a_n, b_n) = [1-1/2^n-1/2^n,1-1/2^n].\)
Illustrating some arguments used in proofs of lineability and spaceability results....

Theorem (Botelho, Favaro, –, Seoane, 2012)

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**Proof.** Let us first consider the following representation of the semi-open interval $[0,1)$ as a disjoint union of intervals:

$$[0,1) = [0,1-1/2) \cup [1-1/2,1-1/4) \cup [1-1/4,1-1/8) \cup \cdots = \bigcup_{n=1}^{\infty} I_n,$$

where $I_n := [a_n, b_n) = [1 - \frac{1}{2^n-1}, 1 - \frac{1}{2^n})$. 
Notice that, for every $n \in \mathbb{N}$ and every $x \in I_n$, there is a unique $x_n \in [0, 1)$ such that
\[ x = (1 - x_n)a_n + x_nb_n. \]
Illustrating some arguments used in proofs of lineability and spaceability results....

Notice that, for every $n \in \mathbb{N}$ and every $x \in l_n$, there is a unique $x_n \in [0, 1)$ such that

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Now, given $p > 0$, let us fix a function $f \in L_p[0, 1] - \bigcup_{q > p} L_q[0, 1]$, and define a sequence of functions $(f_n)_{n=1}^{\infty}$, with $f_n : [0, 1] \rightarrow \mathbb{R}$, as follows:

$$f_n(x) = \begin{cases} f(x_n) & \text{if } x \in l_n, \\ 0 & \text{if } x \notin l_n. \end{cases}$$
Illustrating some arguments used in proofs of lineability and spaceability results....

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So $L_p[0, 1] \setminus \bigcup_{q>p} L_q[0, 1]$ is $\aleph_0$-lineable.
Illustrating some arguments used in proofs of lineability and spaceability results.

To prove that $L_p[0, 1] - \bigcup_{q > p} L_q[0, 1]$ is spaceable the idea is to define a bounded linear and injective operator $T : F \rightarrow L_p[0, 1]$, where $F$ is a Banach space, and such that

$$\overline{T(F)} \cap L_q[0, 1] = \{0\}$$

for every $q > p$. 
Illustrating some arguments used in the proofs of lineability....

It can be proved that

\[ T: \ell_1 \longrightarrow L_p[0, 1], \quad T((\alpha_j)_{j=1}^{\infty}) = \sum_{n=1}^{\infty} \alpha_n f_n \]

is a well defined bounded linear operator.
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This operator is what we need.
Final remarks

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However, and it makes everything more interesting, this technique is limited to certain situations.....

... and new techniques are needed in different situations (in some cases non-constructive approaches).
Final remarks

In our survey we have collected 260 references in some sense related to the subject.... we hope to attract more people to the field...
Thanks...