Non-absolutely convergent integrals with respect to distributions

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For example, to give a sense to the nonabsolutely convergent sum

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we need the ordering of the set of natural numbers, a permutation of the summands can lead to a different result. Similarly, to give a sense to the integral

$$\int_0^{\infty} \frac{\sin x}{x}\, dx$$

and to others nonabsolutely convergent integrals, we need the ordering of the real line.

F is an indefinite "Newton" integral of $f : (a, b) \to \mathbb{R}$ if, for each $x \in (a, b)$,

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where V_g is the variation of g. But what if g is not of finite variation? And what concerning more general integrands

$$\lim_{y \to x} \frac{F(y) - F(x) - U(x, y) - U(x, x)}{???} = 0\ ?$$

Definition (H. Bendová and J.M. 2011)

Let $I = (a, b) \subset \mathbb{R}$ be an interval. We say that $F \colon I \to \mathbb{R}$ is an *indefinite MC-integral* of $f \colon I \to \mathbb{R}$ with respect to $g \colon I \to \mathbb{R}$ if there is a strictly increasing function $\xi \colon I \to \mathbb{R}$ (the so-called *control function*) such that

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It is clear that this integral contains the Newtonian one. But why it is so much better? We can use the variable growth of the control function to override singularities.

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The link to the one-dimensional case:

$$F(b) - F(a) = \mathbf{F}([a, b]), \qquad 0 < a \leq b < \infty.$$

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Advantages: bilipschitz change of variables, integrates all derivatives.

Functional approach to indefinite integral

Idea: Indefinite integral is not

$$A \mapsto \int_A f(x)\,dx$$

where A runs over a family of sets, but

$$\varphi \mapsto \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$$

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"f, regarded as a function" is "f, regarded as a distribution".
Now, nonabsolutely convergent integration means that we can assign a distribution to a function which is not locally Lebesgue integrable.

A version of the functional approach was PU integral: Instead of partition into intervals we can use partitions of unity into functions. This idea is due to J. Jarník and J. Kurzweil (1985, 1988), improved by J. Kurzweil, J. Mawhin, W. Pfeffer (1991).

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General integrands DU; $F(y)-F(x) \sim U(x,y)-U(x,x)$.

Studied e.g. by J. Kurzweil (in connection with GODEs), R. Henstock.

In the functional approach, the general integrand is a variable $x \mapsto \mathcal{G}(x) : \ \Omega \to \mathcal{D}'(\Omega)$. The indefinite integral is a distribution $\mathcal{F}$ such that

$$\mathcal{F}(\varphi) \sim \mathcal{G}(x)(\varphi)$$

if φ is supported in a small neighborhood of x.

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Examples

A Stieltjes integrand $\mathcal{G}(x) = f(x)\mu$

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A varifold type structure: $G_k(\mathbb{R}^n)$ is the Grassmannian manifold, μ is a measure on $\mathbb{R}^n \times G_k(\mathbb{R}^n)$ disintegrated into $(\mu_x)_{x \in \mathbb{R}^n}$:

$$\mathcal{G}(x)(\varphi) = \int_{G_k(\mathbb{R}^n)} \Big(\int_V \varphi(x+y)\, d\mathcal{H}^k(y) \Big)\, d\mu_x(V).$$

Here, the integration of the test function oven the curved surface is replaced by the integration over the tangent space.

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We consider a family $(\mathbf{p}_{x,r})_{x,r}$ of norms on $\mathcal{D} = \mathcal{D}(\Omega)$ related to balls $B(x,r)$, we skip precise assumptions, the model example is

$$\mathbf{p}_{x,r}(\varphi) = r^k \|D^{(k)}\varphi\|_\infty, \qquad \varphi \in \mathcal{D},\ \operatorname{spt}\varphi \subset \overline{B}(x,r).$$

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Here k is a given order of differentiation and $|D^{(k)}\varphi(x)|$ is the k-th order total differential of φ.The dual norm is

$$\mathbf{p}^*_{x,r}(\mathcal{T}) = \sup\Big\{\langle\mathcal{T},\varphi\rangle :\ \varphi \in \mathcal{D}(\mathbb{R}^n),\ \operatorname{spt}\varphi \subset B(x,r),\ \mathbf{p}_{x,r}(\varphi) \le 1\Big\}, \quad \mathcal{T} \in \mathcal{D}'(\mathbb{R}^n).$$

A function $\delta : \Omega \to (0, \infty)$ is termed a *gauge*. A finite system $(B(x_i, r_i))_{i=1}^m$ of balls is called an α*-packing* in Ω if the balls $B(x_i, \alpha r_i)$ are pairwise disjoint and contained in Ω, $i = 1, \dots, m$. Given a gauge δ, we say that the α-packing is δ*-fine* if $r_i < \delta(x_i)$, $i = 1, \dots, m$.

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The purpose of the scaling factor α is to overturn the dependence of the integral integral on the geometry of balls and to enable bilipschitz change of variables.

Definition (J.M; P. Honzík and J.M.)

Let $\mathcal{F}$ be a distribution on Ω and $(\mathcal{G}(x))_{x\in\Omega}$ be a system of distributions on Ω (standard choice $\mathcal{G}(x) = f(x)\mathcal{G}$). We say that $\mathcal{F}$ is an indefinite *packing integral* of $(\mathcal{G}(x))_{x\in\Omega}$ (with respect to the family $(\mathbf{p}_{x,r})_{x,r}$, if there exist $\alpha \geq 1$ such that for each $\varepsilon > 0$ there exists a gauge $\delta : I \longrightarrow (0,\infty)$ such that for each δ-fine α-packing $(B(x_i, r_i)_{i=1}^m$ in Ω we have

$$\sum_{i=1}^{m} \mathbf{p}^*_{x_i,r_i}(\mathcal{F} - \mathcal{G}(x_i)) < \varepsilon.$$

If $\mathcal{G}(x)$ has the form $f(x)\mathcal{G}$ where f is a function and $\mathcal{G}$ a distribution, we say that $\mathcal{F}$ is and *indefinite packing integral of the function f w.r.t. the distribution $\mathcal{G}$*. Then we denote the integral by $\int f\, d\mathcal{G}$. We also use the multiplier notation $f \bullet \mathcal{G} = \int f\, d\mathcal{G}$.

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- The packing integral integrates all derivatives
- The packing integral allows for a bilipschitz change of variables.
- Applications to singular integrals (P. Honzík and J.M.). We extend the class of functions which admit a Riesz transform by means of integration with respect to the singular kernel distribution.

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- We use Vitali type covering argument to find a covering by balls $B(x, r)$ such that $B(x, r/5)$ are disjointed.
- Then we construct a partition of unity related to the selected balls.

Theorem (K. Kuncová and J.M.)

$\mathcal{F}$ is an indefinite packing integral of $(\mathcal{G}(x))_{x\in\Omega}$ w.r.t. $(\mathbf{p}_{x,r})_{x,r}$ if and only if there exist a Radon measure μ on Ω and $\alpha \geq 1$ such that

$$\lim_{r\to 0+} \frac{\mathbf{p}^*_{x,r}(\mathcal{F} - \mathcal{G}(x))}{\mu(B(x,\alpha r))} = 0.$$

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The proof use a method invented by M. Csörnyei in a different context.

In a series of papers and books, W. Pfeffer proves a general Gauss-Green theorem

$$\int_{\Omega} \operatorname{div} \mathbf{f}(x)\, dx = \int_{\partial\Omega} \mathbf{f} \cdot d\boldsymbol{\nu}.$$

The indefinite integral in his work is a function of BV set, a charge.
Functional approach to charges: T. De Pauw and W. Pfeffer 2008, T. De Pauw, R. Hardt, L. Moonens,...

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Now, we know how to integrate with respect to distributions and can to handle integrals with respect to $\boldsymbol{\nu}$ even if $\boldsymbol{\nu}$ is not a measure.

Our aim is to prove that

$$\int \mathbf{f}\, d(D\chi_\Omega) + \int \chi_\Omega\, d(\operatorname{Div} \mathbf{f}) = \operatorname{Div} \int \mathbf{f}\chi_\Omega$$

or in the multiplier notation

$$\mathbf{f} \bullet D\chi_\Omega + \chi_\Omega \bullet \operatorname{Div} \mathbf{f} = \mathbf{f}\chi_\Omega \bullet \mathcal{L}$$

where $\mathcal{L}$ is the lebesgue measure regarded as distribution.
The most powerful versions of the Gauss-Green theorem are fairly complicated. We present here a simple corollary, in which Ω need not be of finite perimeter..

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$$\int \mathbf{f}\, d(D\chi_\Omega) + \int \chi_\Omega\, d(\operatorname{Div} \mathbf{f}) = \operatorname{Div} \int \mathbf{f}\chi_\Omega$$

or in the multiplier notation

$$\mathbf{f} \bullet D\chi_\Omega + \chi_\Omega \bullet \operatorname{Div} \mathbf{f} = \mathbf{f}\chi_\Omega \bullet \mathcal{L}$$

where $\mathcal{L}$ is the lebesgue measure regarded as distribution.
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Theorem (J.M.)

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a countably $(n-1)$-rectifiable boundary. Let $\mathbf{f} \in C(\overline{\Omega}) \cap W^{1,1}(\Omega)$ be a vector field. Then

$$\int_\Omega \operatorname{div} \mathbf{f}(x)\, dx = \int_{\partial\Omega} \mathbf{f} \cdot d\boldsymbol{\nu},$$

where the integral on the right is understood as the packing integral of $\mathbf{f}$ with respect to $\boldsymbol{\nu} = -D\chi_\Omega$.

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Stokes theorem?

Stokes theorem?

- in the euclidean setting, a notable version is not yet established.
- may build on recent developments in GMT of currents, chains, cochains, charges,.. J. Harrison; T. De Pauw, L. Moonens and W. Pfeffer 2009; T. De Pauw and R. Hardt 201x.

Integration in metric spaces

The generalization to metric spaces is a joint work with Kristýna Kuncová. Instead of smooth test functions we use Lipschitz test functions and norms of order at most 1, The Lipschitz norm seems to be the most natural choice, so that we set

$$\mathbf{p}_{x,r}(\varphi) = \|\varphi\|_\infty + r\|\varphi\|_{\mathrm{Lip}}.$$

(One can regard this as a model case and generalize to other norms. e.g. Sobolev type norms on metric spaces.)

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Dual objects to Lipschitz functions (weak* continuous functionals on compactly supported Lipschitz functions) are called *metric distributions*. We integrate families of metric distributions and the indefinite integral is a metric distribution.

Features

- If the space admits a doubling measure, then the packing integral is well defined.

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- The result on the measure control is also available.
- The Lebesgue integral with respect to a Radon measure μ is included.
- The packing integral allows for a bilipschitz change of variables.

Gauss-Green-Stokes theorems

We need the setting of currents on metric spaces: L. Ambrosio and B. Kirchheim 2000, related work T. De Pauw, R. Hardt,...
Roughly, a metric 1-current is a bilinear weak* continuous functional $\mathcal{T}$ on "forms" $\varphi\, d\psi$, where (φ, ψ) is a pair of Lipschitz functions with compact $\operatorname{spt} \varphi\psi$. The definition also requires: if ψ is constant on $\{\varphi \neq 0\}$, then $\mathcal{T}(\varphi\, d\psi) = 0$.
The boundary of a metric 1-current $\mathcal{T}$ is the metric distribution

$$\partial\mathcal{T} : \ \varphi \mapsto \mathcal{T}(1\, d\varphi).$$

We are interested in validity of the product rule

$$\partial(\mathbf{fg} \bullet \mathcal{T}) = \mathbf{f} \bullet \partial(\mathbf{g} \bullet \mathcal{T}) + \mathbf{g} \bullet \partial(\mathbf{f} \bullet \mathcal{T}).$$

Here the currents can be vector valued. For $\mathbf{g} = \chi_\Omega$ we obtain the Gauss-Green-Stokes formula. Indeed, let us observe the "translation table', on the right we have the classical case of the divergence theorem.

A 1-current $\mathcal{T}$ is given	$\mathcal{T}(\varphi\, d\psi) = \int_{\mathbb{R}^n} \varphi \nabla \psi\, dx$
$\mathbf{f} \bullet \mathcal{T}(\varphi\, d\psi)$	$\int_{\mathbb{R}^n} f \varphi \nabla \psi\, dx$
$\partial(\mathbf{f} \bullet \mathcal{T})(\varphi)$	$\int_{\mathbb{R}^n} f \nabla \varphi = -\int_{\mathbb{R}^n} \varphi \operatorname{div} f\, dx$
$\chi_\Omega \bullet \partial(\mathbf{f} \bullet \mathcal{T})(\varphi)$	$-\int_\Omega \varphi \operatorname{div} f\, dx$
$\chi_\Omega \bullet \mathcal{T}(\varphi\, d\psi)$	$\int_\Omega \varphi \nabla \psi\, dx$
$\partial(\chi_\Omega \bullet \mathcal{T})(\varphi)$	$\int_\Omega \nabla \varphi\, dx = \int_{\partial\Omega} \varphi \cdot \boldsymbol{\nu}\, d\mathcal{H}^{n-1}$
$\mathbf{f} \bullet \partial(\chi_\Omega \bullet \mathcal{T})(\varphi)$	$\int_{\partial\Omega} f \varphi \cdot \boldsymbol{\nu}\, d\mathcal{H}^{n-1}$
$\partial(\mathbf{f}\chi_\Omega \bullet \mathcal{T})(\varphi)$	$-\int_\Omega f \cdot \nabla \varphi\, dx$

At the end, in the classical model the formula

$$\partial(\mathbf{f}\chi_\Omega \bullet \mathcal{T}) = \mathbf{f} \bullet \partial(\chi_\Omega \bullet \mathcal{T}) + \chi_\Omega \bullet \partial(\mathbf{f} \bullet \mathcal{T})$$

gives

$$-\int_\Omega \mathbf{f} \cdot \nabla\varphi \, dx = \int_{\partial\Omega} \varphi \mathbf{f} \cdot \, d\boldsymbol{\nu} - \int_\Omega \varphi \operatorname{div} \mathbf{f} \, dx.$$

If the test function equals 1 on $\overline{\Omega}$, we obtain

$$\int_{\partial\Omega} \mathbf{f} \cdot d\boldsymbol{\nu} = \int_\Omega \operatorname{div} \mathbf{f} \, dx.$$

We are able to prove the product rule

$$\partial(\mathbf{f}\mathbf{g} \bullet \mathcal{T}) = \mathbf{f} \bullet \partial(\mathbf{g} \bullet \mathcal{T}) + \mathbf{g} \bullet \partial(\mathbf{f} \bullet \mathcal{T})$$

under some complicated technical assumptions, which however are not as restrictive as they look like.