After the Bishop-Phelps Theorem

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Plan of talk

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  (**Lindenstrauss**' contribution)

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1) Background & History

Definition

(**Victor Klee**, 1958) Let $C \subset E$ be a convex subset of a real locally convex space E. A point x in the boundary of C is a *support point* if $\exists \varphi \in E^*$ such that $\varphi(x) = \sup \varphi(C)$ (and then φ is a *support functional*.)

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Klee asked if every bounded closed subset $C \subset X$, where X is a Banach space, has at least one support point.

Theorem (**R. C. James**, 1957, 1963) A Banach space X is *reflexive* (that is, $X \approxeq X^{**}$) if and only if $\forall \varphi \in X^*, \exists x \in \overline{B_X}$ such that $\varphi(x) = \|\varphi\|$.

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In other words, in reflexive spaces X, every norm one point of $\overline{B_X}$ is a support point for $\overline{B_X}$ and every norm one functional in X^* is a support functional.

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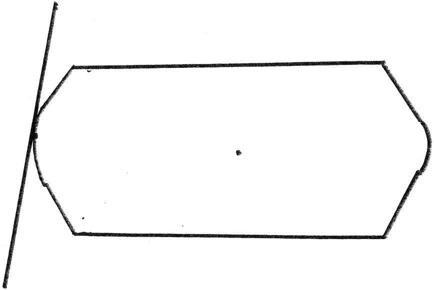
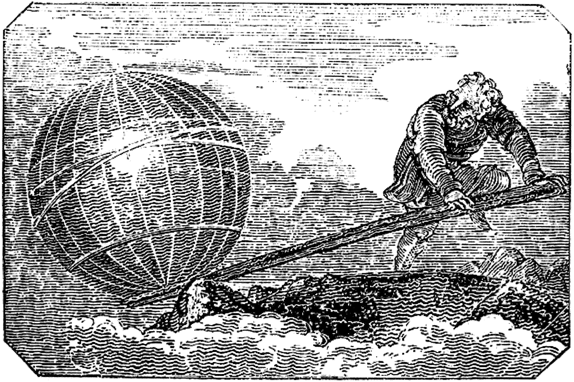
Remark V. Lomonosov (2000) showed that, in general, in a *complex* Banach space, there are closed bounded convex sets with no support points.





2) Bishop-Phelps Theorem; and (a) Bollobás' contribution

Bishop-Phelps Theorem Let X be a Banach space, let $\varphi \in X^*$, $\|\varphi\| = 1$, and let $\epsilon > 0$ be arbitrary. Then $\exists \theta \in X^*$, $\|\theta\| = 1$, with the following two properties:

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(The paper of Bishop and Phelps is 1-1/2 pages long.)

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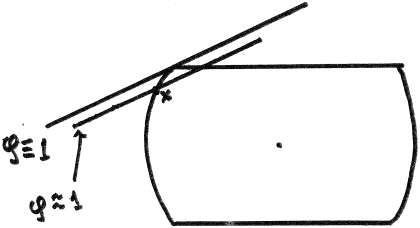
This is known as the **Bishop-Phelps-Bollobás Theorem** (or BPB). Bollobás' paper is also 1-1/2 pages.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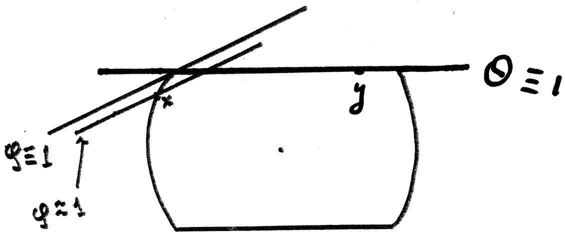
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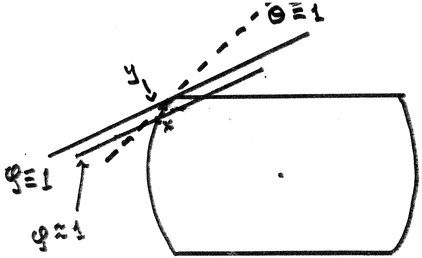
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Examples

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Examples

- (i). If X is reflexive, then we have the exact analogy of the Bishop-Phelps Theorem:
  $\forall T \in \mathcal{L}(X, Y), \forall \epsilon > 0, \exists S \in \mathcal{L}(X, Y)$ *such that* $\|S - T\| < \epsilon$ *and S attains its norm.*

- (ii). **Fact**: $\exists$ an isomorphic copy $Y \backsimeq c_0$ such that Y is strictly convex. Suppose $T : c_0 \to Y$ is an isomorphism. Then Then, T cannot be approximated by a norm-attaining $S \in \mathcal{L}(c_0, Y)$.

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Let X be a Banach space and $n \in \mathbb{N}$. Denote by $\mathcal{L}(^nX)$ the space of continuous $n-$linear functions

$$A : X \times ... \times X \to \mathbb{K},$$

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with norm $\|A\| = \sup\{|A(x_1, ..., x_n) \mid \|x_j\| \leq 1 \text{ for all } j = 1, ...n\}$. Let $\mathcal{L}_{NA}(^n X)$ be the subset of $\mathcal{L}(^n X)$ consisting of all those A whose norm is attained.

First basic question: Is there a Bishop-Phelps theorem for $n-$linear forms? That is, *Given* $A \in \mathcal{L}(^n X)$ *and* $\epsilon > 0$, *is there* $C \in \mathcal{L}_{NA}(^n X)$ *such that* $\|A - C\| < \epsilon$?

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Answer 2: NO, for certain (strange) Banach spaces like the Gowers space (Acosta-Aguirre-Payá, 1996), and also for some not-so-strange Banach spaces like $L_1[0, 1]$ (Choi, 1997).

Special, somewhat easier bilinear ($n = 2$) case: For $A : X \times X \to \mathbb{K}$, let $T_A : X \to X^*, \quad T_A(x_1)(x_2) \equiv A(x_1, x_2)$. In some situations, we can use Lindenstrauss' results to help.

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Theorem (Finet - Payá, 1998) Every operator $L_1 \to L_\infty$ can be approximated by norm-attaining operators.

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Question (b). Is there a Bishop-Phelps theorem for such $n-$homogeneous polynomials?

(ii) Extensions of the Bishop-Phelps-Bollobás Theorem

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Definition A pair of Banach spaces (X, Y) has the *Bishop-Phelps-Bollobás property* if for all $T \in L(X, Y)$ (WLOG, $\|T\| = 1$) and all $\epsilon > 0$, there is $\eta(\epsilon) > 0$ such that if $x_0 \in X, \|x_0\| = 1$ is such that $\|T(x_0)\| > 1 - \eta(\epsilon)$, then there are $S \in L(X, Y)$, $\|S\| = 1$, and $x_1 \in X, \|x_1\| = 1$, that satisfy the following properties:

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$\exists$ necessary and sufficient geometric conditions on Y for (ℓ_1, Y) to have the Bishop-Phelps-Bollobás property.

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Questions: (1). Does $Y = \mathbb{R}^2$ with the Euclidean norm have B?
(2). For arbitrary X and Y, consider the set of norm-attaining operators in $L(X, Y)$. Does this set always contain an infinite dimensional vector space? Does it contain a 2−dimensional vector space? Unknown even for $Y = \mathbb{R}$.